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ABSTRACT

This fifth unit in the SMSG junior high mathematics series is a student text covering the following topics: rational numbers and coordinates; equations; scientific notation, decimals, and the metric system; constructions, congruent triangles, and the Pythagorean property; relative error; and real numbers. (DT)

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STUDENT'S TEXT UNIT NO. 5

ED130874

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MATHEMATICS FOR JUNIOR HIGH SCHOOL
VOLUME 2

PART I

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SCHOOL MATHEMATICS STUDY GROUP

2

YALE UNIVERSITY PRESS

School Mathematics Study Group

Mathematics for Junior High School, Volume 2

Unit 5

Mathematics for Junior High School, Volume 2

Student's Text, Part I

Prepared under the supervision of the
Panel on Seventh and Eighth Grades
of the School Mathematics Study Group:

R. D. Anderson	Louisiana State University
J. A. Brown	University of Delaware
Lenore John	University of Chicago
B. W. Jones	University of Colorado
P. S. Jones	University of Michigan
J. R. Mayor	American Association for the Advancement of Science
P. C. Rosenbloom	University of Minnesota
Veryl Schult	Supervisor of Mathematics, Washington, D.C.

New Haven and London, Yale University Press

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Science Foundation.

FOREWORD

The increasing contribution of mathematics to the culture of the modern world, as well as its importance as a vital part of scientific and humanistic education, has made it essential that the mathematics in our schools be both well selected and well taught.

With this in mind, the various mathematical organizations in the United States cooperated in the formation of the School Mathematics Study Group (SMSG). SMSG includes college and university mathematicians, teachers of mathematics at all levels, experts in education, and representatives of science and technology. The general objective of SMSG is the improvement of the teaching of mathematics in the schools of this country. The National Science Foundation has provided substantial funds for the support of this endeavor.

One of the prerequisites for the improvement of the teaching of mathematics in our schools is an improved curriculum--one which takes account of the increasing use of mathematics in science and technology and in other areas of knowledge and at the same time one which reflects recent advances in mathematics itself. One of the first projects undertaken by SMSG was to enlist a group of outstanding mathematicians and mathematics teachers to prepare a series of textbooks which would illustrate such an improved curriculum.

The professional mathematicians in SMSG believe that the mathematics presented in this text is valuable for all well-educated citizens in our society to know and that it is important for the precollege student to learn in preparation for advanced work in the field. At the same time, teachers in SMSG believe that it is presented in such a form that it can be readily grasped by students.

In most instances the material will have a familiar note, but the presentation and the point of view will be different. Some material will be entirely new to the traditional curriculum. This is as it should be, for mathematics is a living and an ever-growing subject, and not a dead and frozen product of antiquity. This healthy fusion of the old and the new should lead students to a better understanding of the basic concepts and structure of mathematics and provide a firmer foundation for understanding and use of mathematics in a scientific society.

It is not intended that this book be regarded as the only definitive way of presenting good mathematics to students at this level. Instead, it should be thought of as a sample of the kind of improved curriculum that we need and as a source of suggestions for the authors of commercial textbooks. It is sincerely hoped that these texts will lead the way toward inspiring a more meaningful teaching of Mathematics, the Queen and Servant of the Sciences.

The preliminary edition of this volume was prepared at a writing session held at the University of Michigan during the summer of 1959. Revisions were prepared at Stanford University in the summer of 1960, taking into account the classroom experience with the preliminary edition during the academic year 1959-60. This edition was prepared at Yale University in the summer of 1961, again taking into account the classroom experience with the Stanford edition during the academic year 1960-61.

The following is a list of all those who have participated in the preparation of this volume.

R.D. Anderson, Louisiana State University
B.H. Arnold, Oregon State College
J.A. Brown, University of Delaware
Kenneth E. Brown, U.S. Office of Education
Mildred B. Cole, K.D. Waldo Junior High School, Aurora, Illinois
B.H. Colvin, Boeing Scientific Research Laboratories
J.A. Cooley, University of Tennessee
Richard Dean, California Institute of Technology
H.M. Gehman, University of Buffalo
L. Roland Genise, Brentwood Junior High School, Brentwood, New York
E. Glenadine Gibb, Iowa State Teachers College
Richard Good, University of Maryland
Alice Hach, Ann Arbor Public Schools, Ann Arbor, Michigan
S.B. Jackson, University of Maryland
Lenore John, University High School, University of Chicago
B.W. Jones, University of Colorado
P.S. Jones, University of Michigan
Houston Karnes, Louisiana State University
Mildred Keiffer, Cincinnati Public Schools, Cincinnati, Ohio
Nick Lovdjieff, Anthony Junior High School, Minneapolis, Minnesota
J.R. Mayor, American Association for the Advancement of Science
Sheldon Meyers, Educational Testing Service
Muriel Mills, Hill Junior High School, Denver, Colorado
P.C. Rosenbloom, University of Minnesota
Elizabeth Roudebush Mitchell, Seattle Public Schools, Seattle, Washington
Veryl Schult, Washington Public Schools, Washington, D.C.
George Schaefer, Alexis I. DuPont High School, Wilmington, Delaware
Allen Shields, University of Michigan
Rothwell Stephens, Knox College
John Wagner, School Mathematics Study Group, New Haven, Connecticut
Ray Walch, Westport Public Schools, Westport, Connecticut
G.C. Webber, University of Delaware
A.B. Willcox, Amherst College

CONTENTS

FOREWORD v

PREFACE ix

Chapter

1. RATIONAL NUMBERS AND COORDINATES 1
   - 1- 1. The Number Line 1
   - 1- 2. Negative Rational Numbers 7
   - 1- 3. Addition of Rational Numbers 12
   - 1- 4. Coordinates 21
   - 1- 5. Graphs 30
   - 1- 6. Multiplication of Rational Numbers 34
   - 1- 7. Division of Rational Numbers 42
   - 1- 8. Subtraction of Rational Numbers 46

2. EQUATIONS 51
   - 2- 1. Writing Number Phrases 51
   - 2- 2. Writing Number Sentences 58
   - 2- 3. Finding Solution Sets 75
   - 2- 4. Solving Inequalities 91
   - 2- 5. Number Sentences with Two Unknowns 93

3. SCIENTIFIC NOTATION, DECIMALS, AND THE METRIC SYSTEM . 111
   - 3- 1. Large Numbers and Scientific Notation 111
   - 3- 2. Calculating with Large Numbers 117
   - 3- 3. Calculating with Small Numbers 120
   - 3- 4. Multiplication: Large and Small Numbers . . 125
   - 3- 5. Division: Large and Small Numbers 128
   - 3- 6. Use of Exponents in Multiplying and Dividing Decimals 133
   - 3- 7. The Metric System; Metric Units of Length . . 137
   - 3- 8. Metric Units of Area 145
   - 3- 9. Metric Units of Volume 148
   - 3-10. Metric Units of Mass and Capacity 150

4. CONSTRUCTIONS, CONGRUENT TRIANGLES, AND THE PYTHAGOREAN PROPERTY 157
   - 4- 1. Introduction to Mathematical Drawings and Constructions 157
   - 4- 2. Basic Constructions 162
   - 4- 3. Symmetry 172
   - 4- 4. Congruent Triangles 176
   - 4- 5. Showing Two Triangles To Be Congruent 184
   - 4- 6. The Right Triangle 192
   - 4- 7. One Proof of the Pythagorean Property 202
   - 4- 8. Quadrilaterals 206
   - 4- 9. Solids 208

Chapter

5. RELATIVE ERROR 215
  - 5- 1. Greatest Possible Error 215
  - 5- 2. Precision and Significant Digits 219
  - 5- 3. Relative Error, Accuracy, and Percent of Error 223
  - 5- 4. Adding and Subtracting Measures 227
  - 5- 5. Multiplying and Dividing Measures 229

6. REAL NUMBERS 235
  - 6- 1. Review of Rational Numbers 235
  - 6- 2. Density of Rational Numbers 240
  - 6- 3. Decimal Representations for the Rational Numbers 245
  - 6- 4. The Rational Number Corresponding to a Periodic Decimal 250
  - 6- 5. Rational Points on the Number Line 255
  - 6- 6. Irrational Numbers 257
  - 6- 7. A Decimal Representation for $\sqrt{2}$ 262
  - 6- 8. Irrational Numbers and the Real Number System 267
  - 6- 9. Geometric Properties of the Real Number Line. 276

INDEX . . . . following 279

PREFACE

Key ideas of junior high school mathematics emphasized in this text are: structure of arithmetic from an algebraic viewpoint; the real number system as a progressing development; metric and non-metric relations in geometry. Throughout the materials these ideas are associated with their applications. Important at this level are experience with and appreciation of abstract concepts, the role of definition, development of precise vocabulary and thought, experimentation, and proof. Substantial progress can be made on these concepts in the junior high school.

Fourteen experimental units for use in the seventh and eighth grades were written in the summer of 1958 and tried out by approximately 100 teachers in 12 centers in various parts of the country in the school year 1958-59. On the basis of teacher evaluations these units were revised during the summer of 1959 and, with a number of new units, were made a part of sample textbooks for grade 7 and a book of experimental units for grade 8. In the school year 1959-60, these seventh and eighth grade books were used by about 175 teachers in many parts of the country, and then further revised in the summer of 1960. Again during the year 1960-61, this text for grade 8 was used by nearly 200 classes in all parts of the country, and then this edition was prepared in the summer of 1961.

Mathematics is fascinating to many persons because of its opportunities for creation and discovery as well as for its utility. It is continuously and rapidly growing under the prodding of both intellectual curiosity and practical applications. Even junior high school students may formulate mathematical questions and conjectures which they can test and perhaps settle; they can develop systematic attacks on mathematical problems whether or not the problems have routine or immediately determinable solutions. Recognition of these important factors has played a considerable part in selection of content and method in this text.

We firmly believe mathematics can and should be studied with success and enjoyment. It is our hope that this text may greatly assist all teachers who use it to achieve this highly desirable goal.

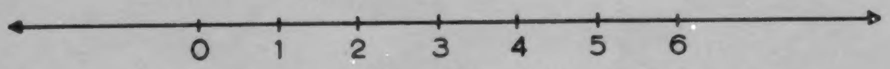
Chapter 1

RATIONAL NUMBERS AND COORDINATES

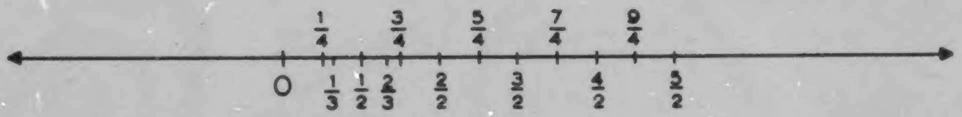
1-1. The Number Line

The idea of number is an abstract one. The development of a good number system required centuries in the civilization of man. To help understand numbers and their uses, many schemes have been used. One of the most successful of these ways to picture numbers is the use of the number line. Suppose we recall the construction of a number line as a starting point for further discussion of numbers.

We think of the line in the drawing below as extending endlessly in each direction. We choose any point of the line and label it 0. Next, choose another point to the right of 0 and label it 1. This determines a unit of length from 0 to 1. Starting at 0, we lay off this unit length repeatedly toward the right on the number line. This determines the location of the points corresponding to the counting numbers 2, 3, 4, 5, ...



The number $\frac{1}{2}$ we associate with the point midway between 0 and 1. By laying off this segment of length one-half unit over and over again, we determine the additional points corresponding to $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, ... Next, by using a length which is one-third of the unit segment and measuring this length successively to the right of zero, we locate the points $\frac{1}{3}$, $\frac{2}{3}$, $\frac{4}{3}$, $\frac{5}{3}$, ... Similarly, we locate the points to the right of 0 on the number line corresponding to fractions having denominators 4, 5, 6, 7, ... Some of these are shown in the following figure.



By this natural process we associate with each rational number a point on the line. Just one point of the line is associated with each rational number. We thus have a one-to-one correspondence between these rational numbers and some of the points of the line. We speak of the point on the number line corresponding to the number 2 as the point 2. Because of this one-to-one correspondence between number and point, we can name each point by the number which labels it. This is one of the great advantages of the number line. It allows us to identify points and numbers and helps us use geometric points to picture relations among numbers. We shall illustrate some of these uses in the next few paragraphs.

Remark. You might think that this one-to-one correspondence assigns a number to *every* point on the line, to the right of 0. This is far from true. In fact, there are many, many more points unlabeled than labeled by this process. These unlabeled points correspond to numbers like π, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, which are not rational numbers. In Chapter 6, we shall study more about such numbers.

Properties of the Number Line

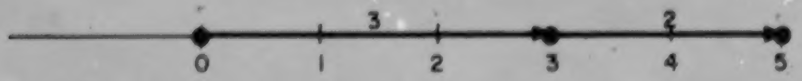
The number line locates numbers by means of points in a very natural way. The construction of the number line locates the rational numbers in order of increasing size. Hence we can always tell where a number belongs on the line. The larger of two numbers always lies to the right. Thus: $5 > 3$ (5 is greater than 3), and on the number line 5 lies to the right of 3. A number greater than 3 corresponds to a point located to the right of 3.

Since $2 < 4$, the point 2 lies to the left of the point 4. We can easily check the relative positions of numbers such as 3, 0, $\frac{5}{4}$, 1, $\frac{4}{3}$, $\frac{3}{2}$. Once we have located the corresponding points on the line, it is easy to tell at a glance whether one number is greater than another or less than another.

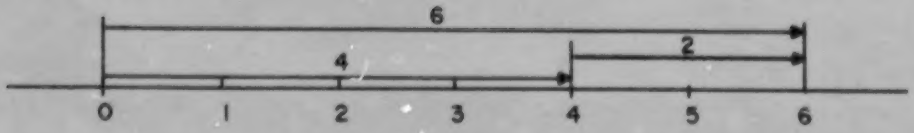
The point corresponding to 0 is chosen as a point of reference and called the *origin*. (Some people call it the fiducial point, and you may call it this to impress your friends, if you wish!) The half-line extending to the right from the origin along the number line is called the positive half-line. Any number which is greater than zero lies on this positive half-line and is called *a positive number*. In particular, we speak of the counting numbers 1, 2, 3, 4 ... as the positive integers. Note that to say a number is positive simply means that it lies to the right of zero on the number line.

Addition on the Number Line

Addition of two numbers can easily be pictured on the number line. To add 2 to 3, we start at 3 and move 2 units to the right. In this way the operation $3 + 2 = 5$ is represented by a motion along the number line. The motion ends at the point corresponding to the sum.



We may also think of the number 3 as determining an arrow (or directed line segment) starting at 0 and ending at 3. To represent the addition of 2 to 3, we simply draw an arrow of length 2 to begin at 3 instead of at 0. The arrow (directed line segment) representing $3 + 2$ thus begins at 0 and ends at 5. To avoid confusion, we frequently indicate these arrows slightly above the number line, as in the following figure for the sum $4 + 2$.

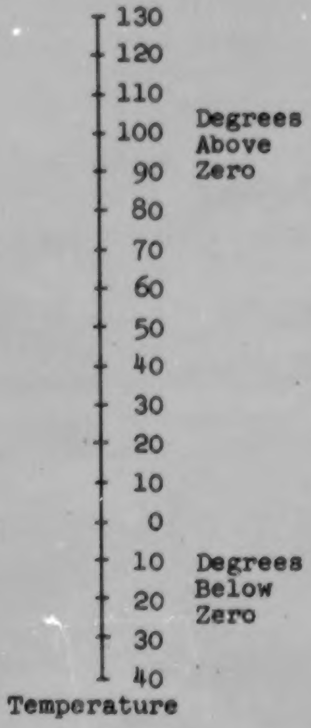

6
2
4
0
1
2
3
4
5
6

The arrows suggest a move of 4 units followed by a move of 2 units in reaching the point (4 + 2). We also interpret the picture as suggesting the addition of two directed line segments of lengths 4 and 2 to form the sum segment of length 6.

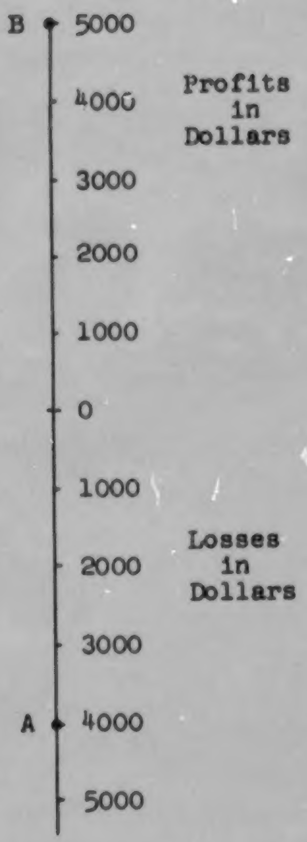
Applications of the Number Line

The number line is used in many forms in our everyday life. A ruler is a fine example, of course. The house numbers along some city streets define a rough form of number line. (If you live on a curved street, you will have to take more mathematics before you study distance and numbering along a curve!)

One of the most common applications of the number line appears in a thermometer. Here we use two number scales, one in each direction along the line. Temperatures above zero appear above zero on the line; temperatures below zero are measured below zero on the line.


130
120
110
100
90
80
70
60
50
40
30
20
10
0
10
20
30
40
Degrees Above Zero
Degrees Below Zero
Temperature

To compare profits and losses of various divisions of a large company for a given month, we might use scales as shown at the right. Division A, which lost 4000 dollars would appear at the point labeled A. Division B, which showed a profit of 5000 dollars, appears at B.


B
5000
4000
3000
2000
1000
0
1000
2000
3000
A
4000
5000
Profits in Dollars
Losses in Dollars

What other examples of the use of number line can you think of?

Exercises 1-1

1. For each of the following numbers draw a number line. Use one inch as the unit of length. Locate a point of origin on the line, and then locate the point corresponding to the number. Just above the number line, draw the corresponding arrow.

(a) 4 (c) $\frac{17}{8}$ (e) 3.75

(b) $\frac{1}{2}$ (d) $2\frac{1}{4}$ (f) 1.125

2. Represent each of the following additions by means of arrows on a number line. Use a separate number line for each addition.

   (a) $2 + 3$

   (b) $1 + 6$

   (c) $\frac{1}{2} + \frac{3}{2}$

   (d) $\frac{3}{8} + \frac{9}{8}$

   (e) $\frac{3}{4} + 1.25$

   (f) $4 + 2\frac{1}{2}$

3. Locate the following numbers on a number line and determine which is the largest in each set.

   (a) $1, \frac{9}{8}, \frac{3}{2}$

   (b) $\frac{3}{4}, \frac{8}{8}, \frac{13}{16}$

   (c) $\frac{13}{4}, 3, \frac{7}{2}$.

4. On some recent automobiles, speedometers use a form of the number line to indicate speed in miles per hour. On these speedometers, a line, similar to an arrow, is used to indicate the speed. Choose a convenient unit of length and construct such a number line showing speeds up to 70 m.p.h. Label on it the points corresponding to the following points. Draw a corresponding arrow for each number just above the number line.

   (a) 10 m.p.h.

   (b) 15 m.p.h.

   (c) 35 m.p.h.

   (d) 60 m.p.h.

5. Locate on a number line the midpoints of the following segments.

   (a) From 0 to 2.

   (b) From $\frac{1}{8}$ to $\frac{5}{8}$.

   (c) From $\frac{1}{2}$ to $\frac{7}{2}$.

   (d) From 2 to 6.5.

6. Use a diagram representing addition by means of arrows on the

number line to show that $2 + 3 = 3 + 2$. What property of addition does this illustrate?

7. Using arrows to represent addition on the number line, show that

$$(2 + 3) + 1 = 2 + (3 + 1).$$

What property of addition does this illustrate?

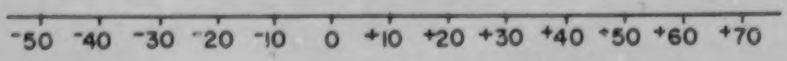
8. Think of a way to represent the product $3 \cdot 2$ by means of arrows on the number line. Try it also for $5 \cdot 2$ and $6 \cdot \frac{1}{4}$.

9. How would you show that $2 \cdot 3 = 3 \cdot 2$ by means of arrows on the number line? What property of multiplication does this illustrate?

1-2. Negative Rational Numbers

In the preceding discussion of number line there is a very serious omission. We did not label the points to the left of zero. We used only the half-line from the origin in the positive direction. To suggest how to label these points (and why we want to!), let us look at the familiar example of temperature.

A number line representing temperature, such as we find on a thermometer, often looks like this.

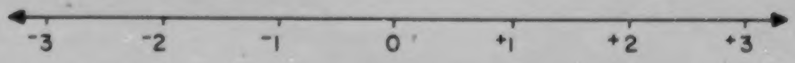


Temperature in Degrees Fahrenheit

Here temperatures less than zero are represented by numbers to the left of the origin and designated by the symbol " $-$ ". Temperatures greater than zero are identified with the sign " $+$ ". Thus, $^{-}10$ refers to a temperature of 10 degrees below zero (to the left of zero), and $^{+}10$ refers to a temperature of 10 degrees above zero (to the right of zero). Actually, above zero and below zero seem more natural terms to use when the scale is vertical.

This idea of distance (or of points) along a line on opposite sides of a fixed point occurs frequently in our ordinary tasks. Think how often we speak of distances to the left or to the right, locations north or south of a given point, altitudes above or below sea level, longitudes east or west, or the time before or after a certain event. In each of these situations, there is a suggestion of points located on opposite sides of a given point (or number), or distances measured in opposite directions from a given point (or number). All of them suggest the need for a number line which uses points to the left of the origin as well as points to the right of the origin.

The natural way to describe such a number line is easy to see. We start with the number line for positive rationals which we have already used. Using the same unit lengths, we measure off distances to the left of zero as shown below:



We locate $^{-}1$ as opposite to $^{+}1$ in the sense that it is 1 unit to the left of zero. Similarly $^{-}2$ is opposite to $^{+}2$, $^{-}(\frac{1}{4})$ is located opposite to $^{+}\frac{1}{4}$, $^{-}(\frac{5}{2})$ is opposite to $^{+}\frac{5}{2}$, etc. These "opposite" numbers, corresponding to points to the left of zero, we call negative numbers. Each negative number lies to the left of zero and corresponds to the opposite positive number. This direction "to the left" is called the negative direction.

We denote negative numbers as $^{-}1$, $^{-}2$, $^{-}(\frac{1}{4})$, $^{-}(\frac{3}{2})$, $^{-}(\frac{9}{8})$ etc., by use of the raised hyphen. We read ($^{-}2$) as "negative two." This negative symbol " $^{-}$ " tells us that the number is less than zero (lies to the left of zero). We sometimes emphasize that a number is positive (greater than zero) by writing the symbol " $^{+}$ "

in a raised position as in $^{+}2$, $^{+}\frac{3}{2}$, etc. Usually we do not do this unless we want to emphasize the positive character of a number.

The new numbers we have introduced by this process are the negative rational numbers. The set consisting of positive rational numbers, negative rational numbers, and zero, we call the rational numbers.

The special set of rational numbers which consists of the positive integers, the negative integers and zero is called the set of integers. We frequently denote this set as:

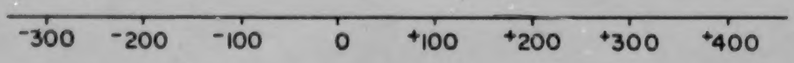
$$I = \{\cdots, {}^{-}4, {}^{-}3, {}^{-}2, {}^{-}1, 0, 1, 2, 3, 4, \cdots\}.$$

Note that the set of integers consists of only the counting numbers and their opposites together with zero.

Examples of the Use of Negative Numbers

The negative numbers are as real and as useful as the positive numbers we have used before. In fact we have used them many times without calling them negative numbers. Their special usefulness is in denoting the idea of "opposite" or "oppositely directed" which we mentioned.

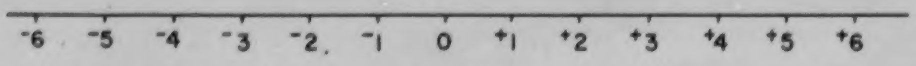
Let us use positive numbers to denote distance east of Chicago. The negative numbers will denote distances west of Chicago. A number line like the one below



Distance from Chicago in miles

can therefore be used to plot the position of an airliner flying an east-west course passing over Chicago. For an airliner flying a north-south course over Chicago, how could you interpret this number line?

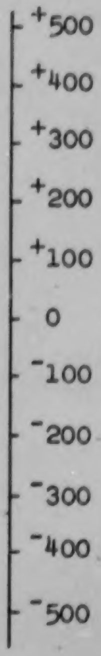
The time before and after the launching of a satellite can be indicated on a number line like the following:


-6
-5
-4
-3
-2
-1
0
+1
+2
+3
+4
+5
+6

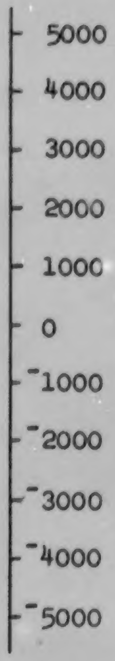
(Seconds before launching) (Seconds after launching)

Note that the number line we use need not be placed horizontally. If we speak of altitude above sea level as positive and altitude below sea level as negative, it may seem more natural to use a number line in the vertical position.

Also, for distances north or south from Kansas City you may wish to use a vertical line.


+500
+400
+300
+200
+100
0
-100
-200
-300
-400
-500

Distance in miles from Kansas City


5000
4000
3000
2000
1000
0
-1000
-2000
-3000
-4000
-5000

Altitude in feet with reference to sea level

To represent business profits and losses, a vertical line is more convenient. A higher position in the line seems naturally to correspond to greater profit. Of course, in other instances some other orientation (not necessarily horizontal or vertical) may seem natural for the number line.

Exercises 1-2

1. Locate on the number line points corresponding to the following numbers.

   (a) $^{-}8$

   (b) $^{-}(\frac{7}{4})$

   (c) $1\frac{3}{4}$

   (d) $\frac{4}{8}$

   (e) 1.5

   (f) $^{-}(\frac{1}{2})$

   Are there any pairs of "opposites" in this list?

2. Sketch the arrows determined by the following rational numbers.

   (a) 6

   (b) $^{-}4$

   (c) $^{-}5.5$

   (d) $\frac{12}{8}$

   (e) $^{-}(\frac{20}{8})$

   (f) $\frac{3}{4}$

3. Arrange the following numbers in the order in which they appear on the number line: $^{-}4$, $\frac{1}{4}$, $^{-}(\frac{7}{4})$, $\frac{5}{8}$, $^{-}6$, $^{-}(\frac{3}{8})$, $\frac{3}{4}$.

   Which is the largest? Which is the smallest?

4. How could you represent the following quantities by means of positive and negative numbers?

   (a) A profit of $2000; a loss of $6000.
   (b) An altitude of 100 ft. above sea level; an altitude of 50 ft. below sea level.
   (c) A loss of 15 yards, a gain of 10 yards.

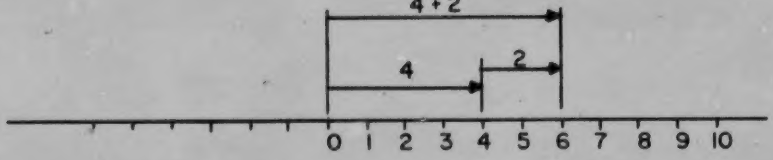
(d) A distance of 2 miles east, a distance of 4 miles west.

5. The elevator control board of a department store lists the floors as B 2, B 1, G, 1, 2, 3. Here G refers to ground level and B 1, B 2 denote basement levels. How could you use positive and negative numbers to label these floors?

6. Draw a number line indicating altitudes from $^{-}1{,}000$ ft. to $^{+}10{,}000$ ft. Use intervals of 1,000 ft. Locate altitudes of $^{-}800$ ft., $^{+}100$ ft., $^{+}2500$ ft., $^{-}500$ ft.

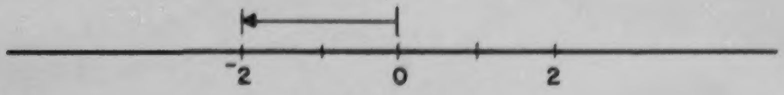
1-3. Addition of Rational Numbers

We saw that the addition of two positive numbers is easily represented on the number line. To refresh your memory, try finding the sums $2 + 4$, $3 + 2$, $1 + 7$ on the number line. On the number line, the sum $4 + 2$ is represented by the point 2 units beyond 4 (or 4 units beyond 2 since it makes no difference which number is chosen first). Note that in adding a positive number to another positive number, we always move to the right (in the positive direction) along the number line. So we describe this process of addition by saying that in adding 2 to 4, we start at 4 and move 2 units to the right, or 2 units in the positive direction. We saw that a convenient way to represent this process is by means of arrows (directed line segments) of appropriate length. Thus, the sum $4 + 2$ corresponds to this picture.



To think of moving from 0 to 4, we draw the directed line segment corresponding to 4. Then beginning at 4, we draw the directed line segment of length 2 which corresponds to 2. In this way we find the arrow of length 6 corresponding to (4 + 2).

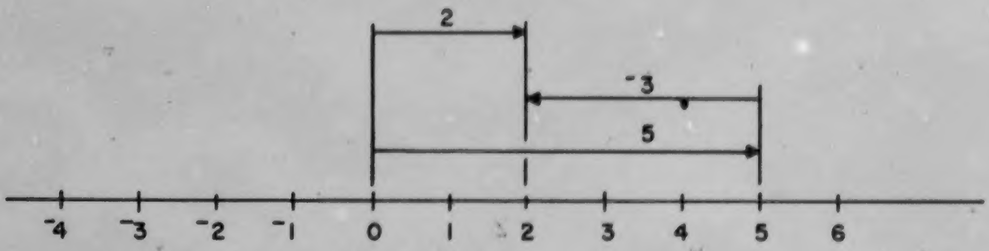
Think now of our construction for the negative numbers in the number line. Remember that $^-2$ is the opposite of 2. To say that $^-2$ is the opposite of 2 means that $^-2$ is the same distance from 0 as 2 but in the negative direction as shown here:



The arrow associated with $^-2$ is 2 units in length and specifies the negative direction as indicated in the sketch. How would you sketch $^-4$; $^-(\frac{1}{2})$; $^-3$; $^-(\frac{5}{3})$?

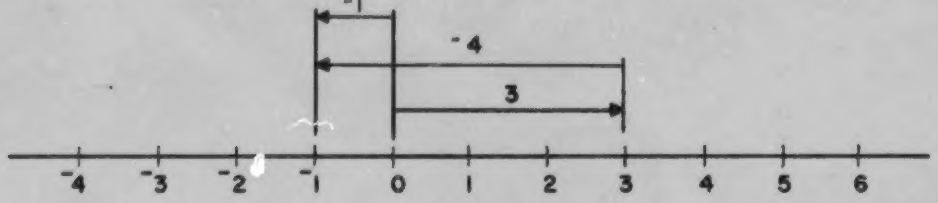
Remark. Many times it is valuable to indicate the approximate position of numbers on the number line in order to compare their locations relative to one another. In such cases only a rough picture is necessary, and careful measurements of length are not justified. We refer to such a rough picture as a sketch.

What would we mean by the sum $5 + (^-3)$? Using directed arrows, we can find the point corresponding to $5 + (^-3)$ by starting at 0, moving 5 units in the positive direction and then 3 units in the negative (opposite) direction. Thus, $5 + (^-3) = 2$, as shown in the following sketch:



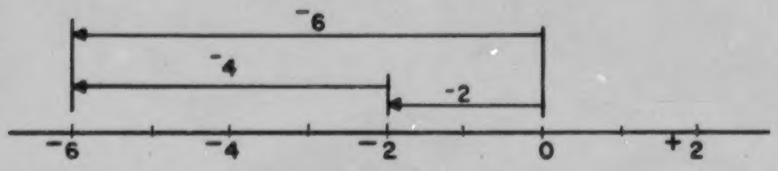
Here, $^{-}3$ is associated with an arrow of length 3 units directed in the negative direction. In adding $(^{-}3)$ to 5, we simply draw the arrow for $^{-}3$ as originating at 5 (that is, beginning at the end of the arrow corresponding to 5).

To add 3 and $^{-}4$, draw a sketch like the following:



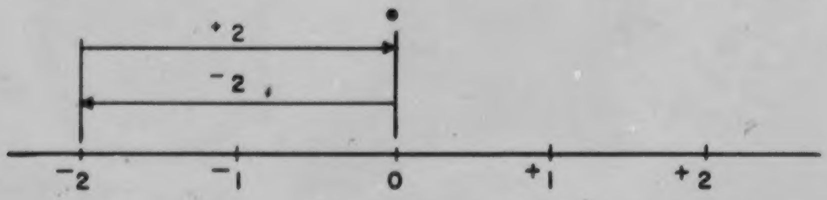
Thus, $3 + (^{-}4) = {}^{-}1$. Find the sum $2 + (^{-}5)$ in the same way.

Consider the sum $(^{-}2) + (^{-}4)$. Here the arrows are both in the negative direction. We see from a sketch that $(^{-}2) + (^{-}4) = (^{-}6)$.



In the same way, find the sums: $(^{-}3) + (^{-}2)$; $(^{-}1) + (^{-}6)$; $(^{-}6) + (^{+}2)$.

One property of special interest is illustrated by the sum $(^{-}2) + (^{+}2)$.



Here $^{-}2$ corresponds to an arrow of length 2 units in the negative direction. Adding $(^{-}2)$ and $(^{+}2)$ corresponds to moving 2 units to the left from 0 and then 2 units back to 0. Thus $(^{-}2) + (2) = 0$. Check that $(^{-}1) + 1 = 0$, $3 + (^{-}3) = 0$, $(^{-}8) + 8 = 0$.

We see by the use of the number line that the addition of numbers, whether positive or negative, is really very simple. We need only keep in mind the location of the numbers on the number line to carry out the operation. We see that:

When both numbers are positive the sum is positive,

as in $\quad ^{+}5 + {}^{+}3 = {}^{+}8;$

and, when both numbers are negative the sum is negative,

as in $\quad (^{-}5) + (^{-}3) = {}^{-}8.$

When one number is positive and one number is negative, it is the number farther from the origin which determines whether the sum is positive or negative.

For example:

In $(^{-}5) + {}^{+}3 = {}^{-}2,$

the sum is negative because the $^{-}5$, which is farther from zero than $^{+}3$, is negative.

In $^{+}5 + (^{-}3) = {}^{+}2,$

the sum is positive, because the $^{+}5$, which is farther from zero than $^{-}3$, is positive.

Another way of saying this is, the arrow of greater length determines whether the sum is positive or negative. In fact, this one rule also works in the case when both numbers are positive or when both numbers are negative. Notice that in cases like $(^-2) + 2$ and $3 + (^-3)$, the arrows are of equal lengths but opposite in direction. In these cases the sum will be zero.

Exercises 1-3a

1. Find the following sums and sketch, using arrows on the number line.

   (a) $9 + (^-5)$

   (b) $10 + (^-7)$

   (c) $(^-8) + 11$

   (d) $5 + (^-10)$

   (e) $(^-12) + 7$

   (f) $3 + (^-11)$

2. Supply the missing numbers in each of the following statements so that each statement will be true.

   (a) $3 + (^-3) = (\quad)$

   (b) $(\quad) + (^-4) = 0$

   (c) $(^+6) + (\quad) = 0$

   (d) $(^-75) + 74 = (\quad)$

   (e) $^-(\frac{1}{2}) + (\quad) = 0$

   (f) $\frac{1}{3} + (\quad) = 1$

   (g) $14 + (^-2) = (\quad)$

   (h) $(^-0.45) + 0.45 = (\quad)$

3. Obtain the sum in each of the following problems.

   (a) $25 + (^-6)$

   (b) $(^-5) + (^-7)$

   (c) $(^-8) + 3$

   (d) $(^-20) + (^-10)$

   (e) $17 + (^-23)$

   (f) $(^-6) + 9$

4. Supply a number in each blank space so that each sum will be correct.

(a) $7 + (\quad) = 2$

(b) $9 + (\quad) = 0$

(c) $10 + (\quad) = {}^{-}1$

(d) $({}^{-}4) + (\quad) = ({}^{-}2)$

(e) $({}^{-}8) + (\quad) = ({}^{-}16)$

(f) $({}^{-}4) + (\quad) = ({}^{-}10)$

5. A company reports income for the first six months of a year as follows:

January	$5000	profit	April	$1000	profit
February	$2000	profit	May	$4000	loss
March	$6000	loss	June	$3000	loss

(a) How could you represent these income figures in terms of positive and negative numbers?

(b) What is the total income for the six-month period?

(c) What is the total income for the first three months of the year?

(d) What is the total income for the four-month period, March, April, May and June?

6. A boy rows upstream at a speed of 4 miles per hour against a current of 2 miles per hour.

(a) How could you use positive and negative numbers in representing these speeds?

(b) What would represent his actual speed upstream?

7. In four successive plays from scrimmage, starting at its own 20 yard line, Franklin High makes

a gain of 17 yards, then
a loss of 6 yards, next
a gain of 11 yards, and finally
a loss of 3 yards.

(a) Represent the gains and losses in terms of positive and negative numbers.
(b) Where is the ball after the fourth play?
(c) What is the net gain after the four plays?

8. An airplane traveling at 13,000 ft. makes a climb of 5000 ft. followed by a descent of 3000 ft.
   (a) Represent the plane's final altitude as a sum of positive and negative numbers.
   (b) What is the plane's altitude after the descent?

9. (a) Think of a way to represent the product $3(^-2)$ by means of arrows on the number line. Try it also for:
   (b) $5 \cdot (^-1)$.
   (c) $2 \cdot {}^-(\frac{1}{4})$.

Inverse Elements under Addition

Recall that $^+2 + (^-2) = 0$. This sentence says that $^-2$ is the number which when added to $^+2$ yields 0. We saw in Volume I that 0 is the identity element under the operation of addition. Any two numbers with sum 0 are said to be inverse elements under addition. Hence $^-2$ is the inverse element corresponding to $^+2$ under the operation of addition. We call $^-2$ the *additive inverse* of $^+2$. Likewise $^+2$ is the additive inverse of $^-2$. Taken together, the elements $^+2$ and $^-2$ are called additive inverses.

Class Exercises 1-3a

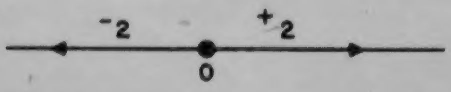
1. Find the additive inverse of each of the following numbers.
   7, $^-9$, 11, $^-12$, $^-6$, 15, $^-20$, 0, $^-(\frac{2}{3})$,
   $\frac{4}{9}$, $^-(\frac{7}{8})$, $\frac{30}{31}$.

2. Which of the following pairs are additive inverses?

   (a) $^{+}20$, $^{+}20$ (c) $.5$, $^{-}(\frac{1}{2})$

   (b) $^{-}5$, $^{-}5$ (d) $^{-}(\frac{1}{2})$, $\frac{2}{4}$

On the number line we see that any number and its additive inverse will be represented by arrows of the same length and opposite direction, as indicated by the sketch of $^{+}2$ and $^{-}2$.



When added, these "opposite" arrows of equal length always give 0. To add $^{+}2$ to $^{-}2$, we think of moving 2 units in the negative direction from 0 and then 2 units back in the positive direction. The two movements are precisely the opposite of each other and bring us back to the starting point. The motions described are true "inverses" of each other, for when one is added to the other, the final result is 0. The number line thus provides a geometric picture for the meaning of additive inverses; the sum of two oppositely directed arrows (motions) of equal length is zero.

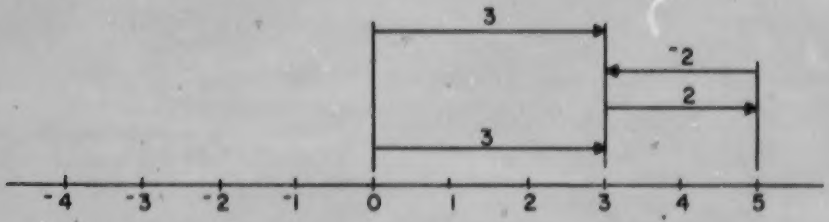
In the addition of a positive and a negative number, we noted that the longer arrow determines whether the sum is positive or negative. The length of the arrow for the sum can be obtained by the picture of additive inverses. For example, in the sum $5 + (^{-}2) = 3$ we may write

$$5 + (^{-}2) = 3 + 2 + (^{-}2)$$

by introducing the additive inverse of the shorter arrow. Then, since $2 + (^{-}2) = 0$ we have

$$5 + (^{-}2) = 3 + 2 + (^{-}2) = 3 + 0 = 3.$$

Note that the other arrow of the two into which 5 is separated represents the sum 3.



This procedure is a general one, as the following examples illustrate.

$$8 + (^-7) = 1 + 7 + (^-7) = 1 + 0 = 1$$

$$(^-8) + 7 = (^-1) + (^-7) + 7 = {^-1} + 0 = {^-1}$$

$$19 + (^-26) = 19 + (^-19) + (^-7) = 0 + (^-7) = {^-7}$$

In each case, the additive inverses add up to zero, and the remaining number is the sum.

Class Exercises 1-3b

1. Sketch the arrows corresponding to the numbers in the above three examples. In each case, determine the arrow corresponding to the sum.
2. Perform the following additions by introducing the additive inverse for the shorter arrow, and sketch the operation on the number line.

   (a) $10 + (^-5)$ (c) $(^-4) + 3$
   (b) $8 + (^-6)$ (d) $(^-13) + 9$

Exercises 1-3b

1. Complete the following:

   (a) $10 + {^-7} = \ ? + 7 + {^-7}$ (d) $23 + {^-18} = \ ? + \ ? + {^-18}$
   (b) ${^-14} + 54 = {^-14} + 14 + \ ?$ (e) ${^-36} + 20 = \ ? + \ ? + 20$
   (c) $12 + {^-14} = 12 + \ ? + {^-2}$

2. Perform the following additions using the additive inverse.
Example: $28 + {}^-20 = 8 + 20 + {}^-20 = 8 + 0 = 8$.

(a) $42 + {}^-12$
(b) ${}^-12 + 8$
(c) ${}^-7 + 35$
(d) $6 + {}^-17$
(e) $344 + {}^-140$
(f) ${}^-172 + 96$

3. State whether the sum will be positive or negative.
Example: ${}^-172 + 37 \longrightarrow$ negative.

(a) $26 + {}^-24$
(b) $72 + {}^-92$
(c) ${}^-376 + 374$
(d) ${}^-4{,}312 + 4{,}324$
(e) $1{,}436{,}312 + {}^-1{,}436{,}310$
(f) $\frac{3}{8} + {}^-(\frac{1}{4})$
(g) ${}^-(\frac{15}{16}) + 0.75$
(h) $\frac{3}{10} + {}^-(\frac{2}{5})$
(i) $\frac{4}{7} + {}^-(\frac{7}{14})$
(j) ${}^-0.132 + 0.0132$
(k) ${}^-3.172 + 3.1724$
(l) $0.0012 + {}^-9$
(m) ${}^-3.025 + 3\frac{1}{4}$
(n) $\frac{5}{12} + {}^-(\frac{7}{16})$

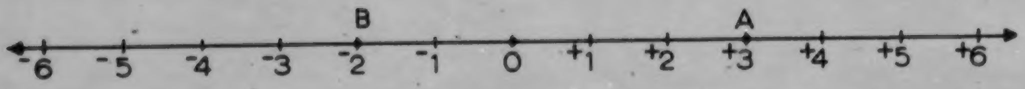
4. We know that ${}^+3 > {}^-17$. Is the following statement true or false? "The sum of a positive number and a negative number always has the sign of the greater number." Explain your answer.

5. The sum of a positive number and a negative number is zero when ________________________.

1-4. Coordinates

Coordinates on a Line

Let us consider the number line from a different point of view. As we have seen, a rational number can always be associated with a point on the number line. The number associated in this way with a point of the line is called a coordinate of the point. In the following drawing, the number zero is associated with the reference point called the origin.



Point A is denoted by the number $(^{+}3)$. Point B is denoted by $(^{-}2)$. We write $A(^{+}3)$ to mean that A is the point with coordinate $^{+}3$. Likewise $B(^{-}2)$ means that B is the point with coordinate $(^{-}2)$ on the line.

Recall that every positive rational number is associated with a point on the positive half-line. Every negative rational number corresponds to a point on the negative half-line. The coordinate we have assigned to a point in this way tells us two things. It tells us the distance from the origin to the point. It also tells us the direction from the origin to the point.

Exercises 1-4a

1. Draw a segment of a number line 6 inches in length. Mark off segments of length one inch and place the origin at its midpoint. On the line locate the following points:

   $A(^{-}1)$, $B(\frac{5}{2})$, $C(1)$, $T(0)$, $L\left({}^{-}(\frac{3}{2})\right)$, $P(^{-}2)$.

2. (a) In Problem 1, how far is it in inches between the point labeled T and the point labeled L?
   (b) between P and B?
   (c) between L and B?
   (d) from the origin to A?

3. Using a number line with 1 inch as the unit of length, mark the following points:

   $R(\frac{1}{3})$, $S(\frac{5}{6})$, $D\left({}^{-}(\frac{3}{2})\right)$, $F(0)$, $E(\frac{3}{2})$.

4. If the line segment in Problem 3 were a highway and if it were drawn to a scale of 1 inch representing 1 mile, how far in miles would it be between these points on the highway:

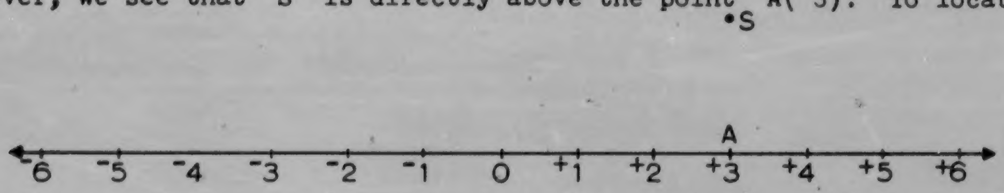
   (a) F and R? (b) D and E?

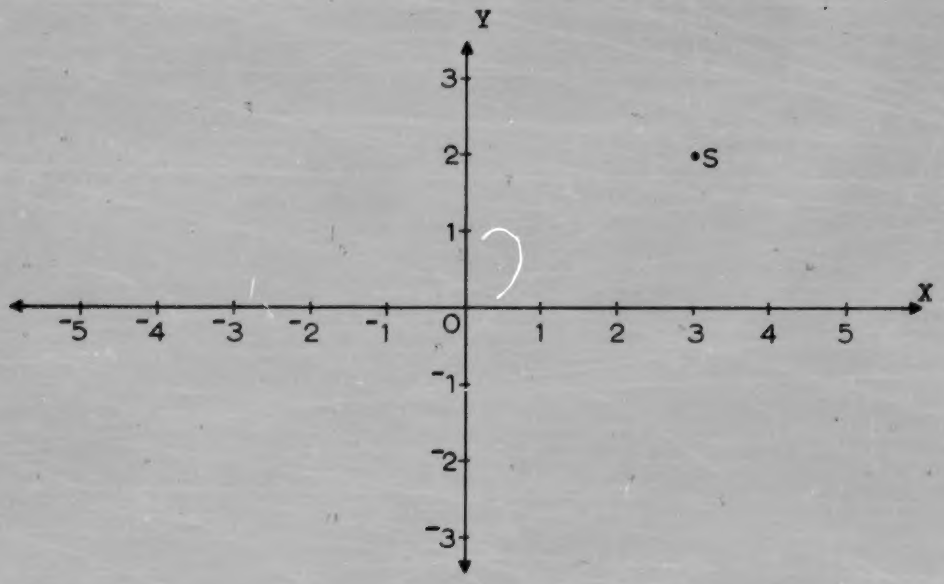
5. Draw a number line in a vertical instead of horizontal position. Mark your number scale with positive numbers above the origin and negative numbers below the origin. Label points to correspond with the rational numbers 0, 1, 2, 3, $^-1$, $^-2$, $^-3$, $^-4$.

Coordinates in the Plane

Recall from your previous work in this chapter that number lines can be drawn vertically as well as horizontally.

You have learned that a single coordinate locates a point on the number line. A point like S in the next figure is not on the number line and cannot be located by a single coordinate. However, we see that S is directly above the point A($^+3$). To locate point S, draw a vertical number line perpendicular to the horizontal number line and intersecting it at the origin. Your drawing should look like this:

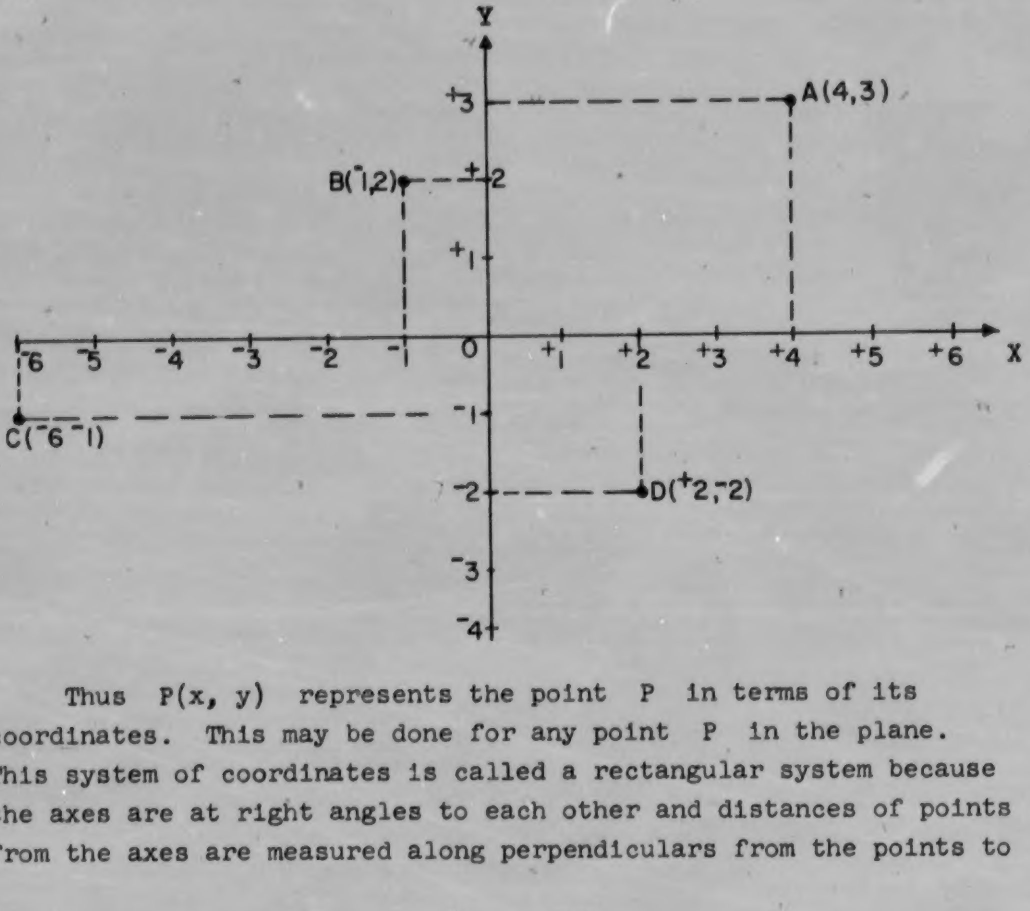




The horizontal number line is called the X-axis and the vertical number line is called the Y-axis. When we refer to both number lines we call them the axes.

To determine the coordinates of point S, draw a line segment from point S perpendicular to the X-axis. It intersects the X-axis at $(^+3)$. Now draw a perpendicular from point S to the Y-axis. It intersects the Y-axis at $(^+2)$. Point S is said to have an x-coordinate of $(^+3)$ and a y-coordinate of $(^+2)$, which we write as $(^+3, {}^+2)$. We use parentheses and always write the x-coordinate before the y-coordinate.

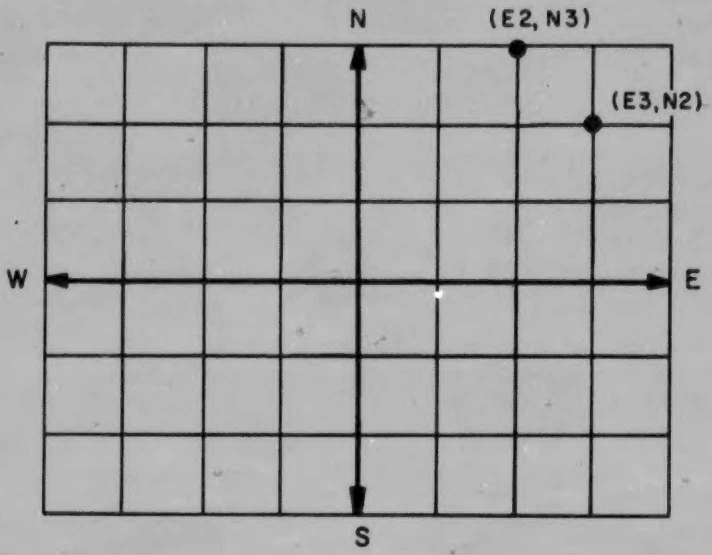
In the diagram below, observe how the coordinates of points A, B, C, and D were located.



Thus P(x, y) represents the point P in terms of its coordinates. This may be done for any point P in the plane. This system of coordinates is called a rectangular system because the axes are at right angles to each other and distances of points from the axes are measured along perpendiculars from the points to the axes. Each ordered pair of rational numbers is assigned to a point in the coordinate plane. Locating and marking the point with respect to the X-axis and the Y-axis is called plotting the point.

The idea of a coordinate system is not new to you. When you locate a point on the earth's surface, you do so by identifying the longitude and latitude of the point. Note that the order in which you write these numbers is important. For example, suppose you looked up the longitude and latitude of your home town and accidentally switched the numbers around. It is possible that your description would place the location of your home town in the middle of the ocean.

Suppose you were giving directions to help a friend locate a certain place in a city laid out in rectangular blocks (streets at right angles to each other). You tell him to start at the center of the city, go 3 blocks east and 2 blocks north (see diagram below). Would this be the same as telling him to go 2 blocks east and 3 blocks north? Of course not! Do you see why it is important to be careful with the order when writing a pair of coordinates?



Exercises 1-4b

1. Given the following set of ordered pairs of rational numbers, locate the points in the plane associated with these pairs.

   $\{(4,1),\ (1,0),\ (0,1),\ (2,4),\ (4,4),$
   $(^-1,^-1),\ (^-3,3),\ (4,^-3),\ (^-5,3),\ (0,^-5),\ (^-6,0)\}.$

2. On squared paper draw a pair of axes and label them. Plot the points in the following sets. Label each point with its coordinates. Use a different pair of axes for each set.

   Set A = $\{(6,^-3),\ (^-7,^-1),\ (^-9,^-7),\ (5,^-1),\ (^-8,10),$
   $(0,0),\ (^-1,^-1),\ (4,3)\}.$

   Set B = $\{(1,1),\ (6,^-5),\ (^-3,^-3),\ (4,^-10),\ (^-9,^-6),$
   $(^-8,0),\ (0,^-5),\ (^-2,^-5)\}.$

3. (a) Plot the points in the following set:
   C = $\{(0,0),\ (^-1,0),\ (^+1,0),\ (^-2,0),\ (^+2,0),$
   $(^-3,0),\ (^+3,0)\}.$

   (b) Do all of the points named in Set C seem to lie on the same line?

   (c) What do you notice about the y-coordinate for each of the points?

   (d) Are there any points on this line for which the y-coordinate is different from zero?

4. (a) Plot the points in the following set:
   D = $\{(0,0),\ (0,^-1),\ (0,^+1),\ (0,^-2),\ (0,^+2),$
   $(0,^-3),\ (0,^+3)\}.$

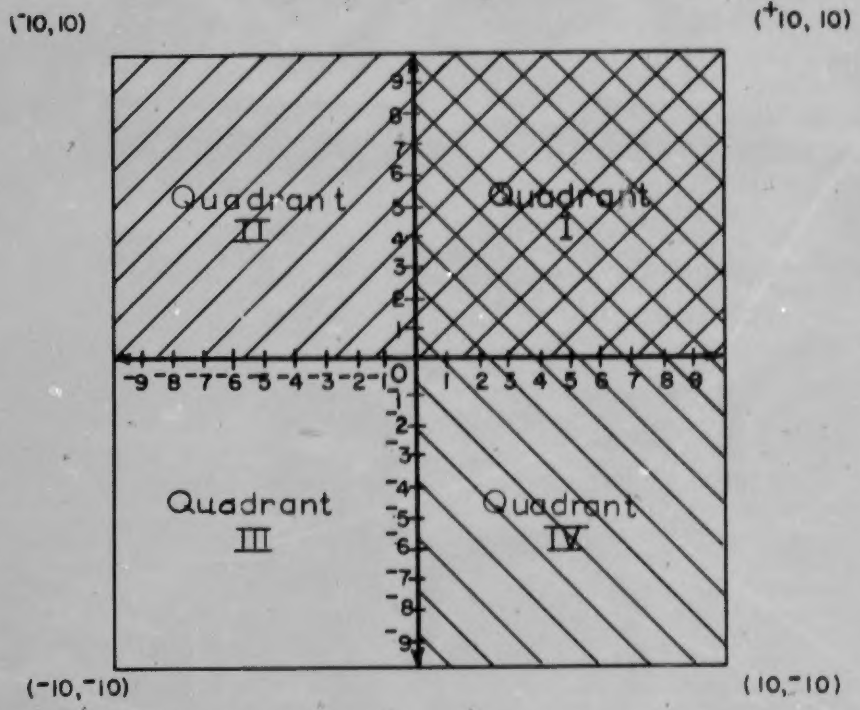
   (b) Do all of the points named in Set D seem to lie on the same line?

   (c) What do you notice about the x-coordinate for each of the points?

   (d) Are there any points on this line for which the x-coordinate is different from zero?

Did you notice that the half planes above and below the X-axis intersect the half planes to the right and to the left of the

Y-axis? These intersections are called quadrants and are numbered in a counter-clockwise direction with Quadrant I being the intersection of the half plane above the X-axis and the half plane to the right of the Y-axis. This quadrant does not include the points on the positive X-axis or positive Y-axis, nor does it include the origin.


(⁻10, 10)
(⁺10, 10)
Quadrant II
Quadrant I
Quadrant III
Quadrant IV
⁻9 ⁻8 ⁻7 ⁻6 ⁻5 ⁻4 ⁻3 ⁻2 ⁻1 0 1 2 3 4 5 6 7 8 9
9 8 7 6 5 4 3 2 1
⁻1 ⁻2 ⁻3 ⁻4 ⁻5 ⁻6 ⁻7 ⁻8 ⁻9
(⁻10, ⁻10)
(10, ⁻10)

Points in the intersection set of these two half planes are in the first quadrant or Quadrant I. The intersection of the half plane above the X-axis and half plane to the left of the Y-axis is Quadrant II. Quadrant III is the intersection of the half plane below the X-axis and the half plane to the left of the Y-axis. Quadrant IV is the intersection of the half plane below the X-axis and the half plane to the right of the Y-axis. Note that the coordinate axes are not a part of any quadrant.

The numbers in ordered pairs may be positive, negative, or zero, as you have noticed in the exercises. Both numbers of the pair may be positive. Both numbers may be negative. One may be positive and the other negative. One may be zero, or both may be zero.

Class Exercises 1-4

1. Given the following ordered pairs of numbers, write the number of the quadrant in which you find the point represented by each of these ordered pairs.

Ordered Pair	Quadrant
(a) $(3, 5)$	______
(b) $(1, {}^-4)$	______
(c) $({}^-4, 4)$	______
(d) $({}^-3, {}^-1)$	______
(e) $(8, 6)$	______
(f) $(7, {}^-1)$	______
(g) $({}^-3, {}^-5)$	______

2. (a) Both numbers of the ordered pair of coordinates are positive. The point is in Quadrant __________.
   (b) Both numbers of the ordered pair of coordinates are negative. The point is in Quadrant __________.
   (c) The x-coordinate of an ordered pair is negative and the y-coordinate is positive. The point is in Quadrant ____.
   (d) The x-coordinate of an ordered pair is positive and the y-coordinate is negative. The point is in Quadrant ____.

3. (a) If the x-coordinate of an ordered pair is zero and the y-coordinate is not zero, where does the point lie?
   (b) If the x-coordinate of an ordered pair is not zero and the y-coordinate is zero, where does the point lie?
   (c) If both coordinates of an ordered pair are zero, where is the point located?

4. Points on either the X-axis or the Y-axis do not lie in any of the four quadrants. Why not?

Exercises 1-4c

1. (a) Plot the points of set $L = \{A(^{+}2,{}^{+}1),\ B(^{+}2,{}^{+}3)\}$.
   (b) Use a straightedge to join A to B. Extend line segment AB.
   (c) Line AB seems to be parallel to which axis?

2. (a) Plot the points of set $M = \{A(^{+}2,{}^{+}3),\ B(^{+}5,{}^{+}3)\}$.
   (b) Use a straightedge to join A to B. Extend line segment AB.
   (c) Line AB seems to be parallel to which axis?

3. (a) Plot the points of set $N = \{A(0,0),\ B(^{+}2,{}^{+}3)\}$.
   (b) Join A to B. Extend line segment AB.
   (c) Is line AB parallel to either axis?

4. (a) Plot the points of set $P = \{A(^{+}4,{}^{+}4),\ B(^{+}2,\ 0)\}$.
   (b) Join A to B. Extend line segment AB.
   (c) Plot the points of set $Q = \{C(^{+}6,{}^{+}3),\ D(0,{}^{+}1)\}$.
   (d) Join C to D. Extend line segment CD.
   (e) What is the intersection set of lines AB and CD?

5. (a) Plot the points of set $R = \{A(0,0),\ B(^{+}6,\ 0),\ C(^{+}3,{}^{+}4)\}$ on the coordinate plane.
   (b) Use a straightedge to join A to B, B to C, C to A.
   (c) Is the triangle (1) scalene, (2) isosceles, or (3) equilateral?

6. (a) Plot the points of set $S = \{A(^{+}2,{}^{+}1),\ B(^{-}2,{}^{+}1),\ C(^{-}2,{}^{-}3),\ D(^{+}2,{}^{-}3)\}$.
   (b) Use a straightedge to join A to B, B to C, C to D, and D to A.
   (c) Is the figure a square?
   (d) Draw the diagonals of the figure.
   (e) The coordinates of the point of intersection of the diagonals seem to be ____________?

7. (a) Plot the points of set $T = \{A(^+2,^+1),\ B(^+3,^+3),\ C(^-2,^+3),\ D(^-3,^+1)\}$.
   (b) Use a straightedge to join A to B, B to C, C to D, and D to A.
   (c) What is the name of the quadrilateral formed?
   (d) Draw the diagonals of quadrilateral ABCD.
   (e) The coordinates of the point of intersection of the diagonals seem to be ___________?

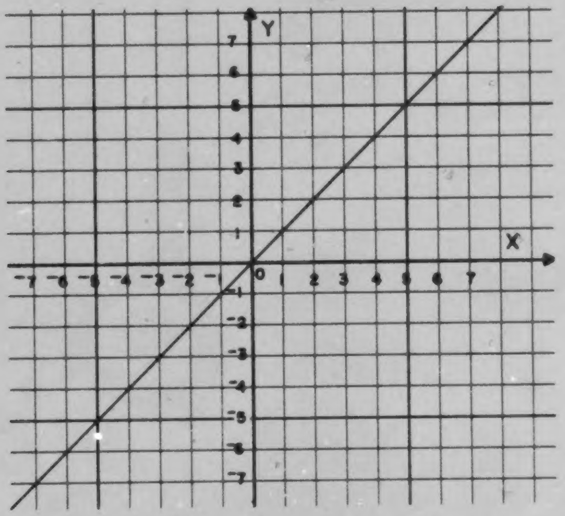
1-5. Graphs

Consider the set of points whose coordinates (x,y) satisfy the condition:

$$y = x.$$

Draw a pair of coordinate axes and label them. Locate these points: (0,0), (1,1), (5,5), $(^-2,^-2)$, $(^-4,^-4)$. The condition y = x is satisfied for each of these points because in each case the y-coordinate is equal to the x-coordinate. Can you find another point in the plane whose y-coordinate is equal to its x-coordinate?

If you have plotted correctly the points listed above, you can draw a line containing them and also containing other points for which y = x.

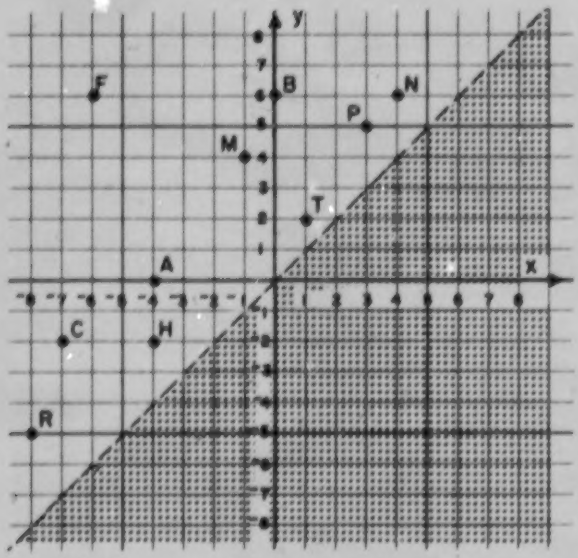


Graph of y = x

Is there any point on this line whose y-coordinate is different from its x-coordinate? The graph of the set of points described by the condition $y = x$ is the line through the origin extending into quadrants I and III and making angles with the coordinate axes that are equal in measure.

Class Exercises 1-5

In these exercises, use the figure to the right.



1. We have just seen that a line contains all the points satisfying the condition $y = x$. Let us now consider the condition $v > x$. The ordered pair (3,5) fits this condition. It is named by what letter in the diagram? Another ordered pair that satisfies $y > x$ is ($^-1$, 4). What letter names this point in the diagram? Some other ordered pairs that fit this condition are: (1,2), (4,6), ($^-4$, 0), (0,6), ($^-6$, 6). What letters are used to name these points?

2. Thus far all the points whose ordered pairs satisfy the condition $y > x$ seem to lie above the line. We should examine the coordinates of points C, H, and R also since they are above the line. Is the ordered pair named by H, ($^-2$,$^-4$) or

$(^-4, ^-2)$? Naturally, you said $(^-4, ^-2)$. Don't forget that this means $x = {^-4}$ and $y = {^-2}$. Do these values satisfy the condition $y > x$? Check to see whether points C and R "belong" to the condition $y > x$. Is there any difference between the condition $y > x$ and the condition $x < y$?

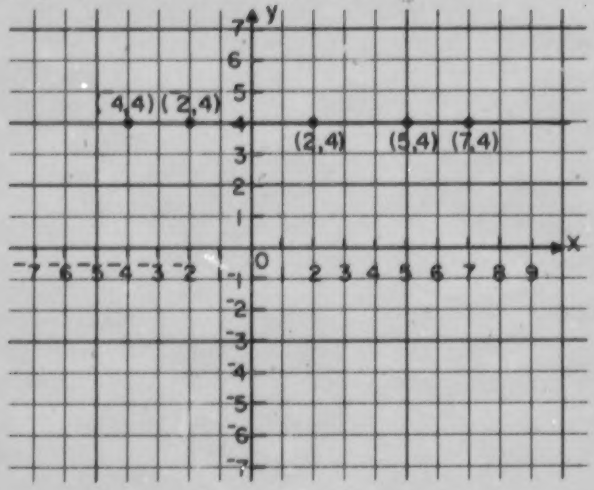
3. You should recall that a line in a plane determines two half-planes. How can this idea be used to describe the set of points whose coordinates satisfy the condition $y > x$?

4. Where do the points whose coordinates satisfy $y < x$ lie? Check by mentally plotting the points associated with the ordered pairs $(1, ^-3)$, $(4, 1)$, $(2, 0)$, $(^-5, ^-2)$, $(4, ^-7)$. Do you find that all of these points lie in the half-plane below the line?

5. Where are the points whose coordinates satisfy $x > y$? Do the points whose coordinates satisfy $x = y$ lie in either half-plane determined by the line? Explain.

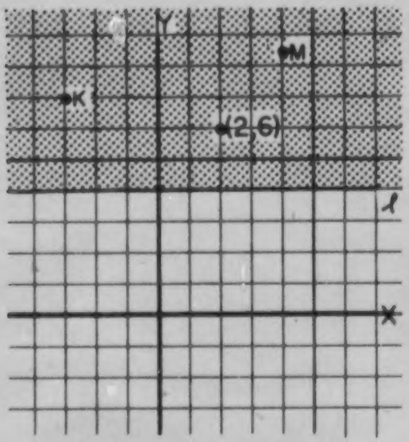
Consider the condition $y = 4$. It is satisfied by these ordered pairs: $(2, 4)$, $(5, 4)$, $(^-2, 4)$, $(7, 4)$, $(^-4, 4)$. The condition is satisfied by any ordered pair whose y-coordinate is 4, regardless of the value of its x-coordinate.

The graph of the set of points described by the condition $y = 4$ is the line parallel to the X-axis and 4 units above it.

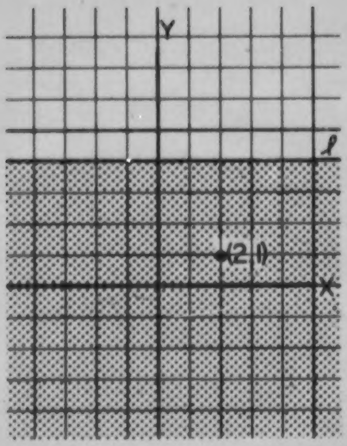


Now let us consider these conditions:

(a) $y > 4$



(b) $y < 4$



In diagram (a) line ℓ is the graph of the set of points described by the condition $y = 4$. Choose a point $(2,6)$. Does the condition $y = 4$ describe this ordered pair? Since the y-coordinate in this ordered pair is greater than 4, the condition $y > 4$ describes it. Are there other points in the plane with y-coordinates greater than 4? Locate two other points, K and M, with y-coordinates greater than 4. Are these points above the line ℓ ? Yes, they are in the shaded region which is one of the half-planes determined by the line $y = 4$.

The graph of the set of points described by the condition $y > 4$ lies in the half-plane above the line 4 units above and parallel to the X-axis. The line is not part of the graph.

In the shaded part of diagram (b) are located points for which the coordinates satisfy the condition $y < 4$. Locate point $(2,1)$ in this region. Since the y-coordinate is less than 4, the condition $y < 4$ describes it. Locate other points in the plane with y-coordinates less than 4. Are these points in the half plane below line ℓ ? Try other points in the half-plane below line ℓ to see if they satisfy the condition $y < 4$.

The graph of the set of points described by the condition $y < 4$ lies in the half-plane below the line 4 units above and parallel to the X-axis.

Exercises 1-5

1. Sketch the graph of the set of points selected by each condition below. Use different coordinate axes for each graph.

   (a) $y = {}^{+}2$
   (b) $y > {}^{+}2$
   (c) $y < {}^{+}2$
   (d) $x = 3$
   (e) $x > 3$
   (f) $x < 3$
   (g) $x = {}^{-}3$
   (h) $x > {}^{-}3$
   (i) $x < {}^{-}3$
   (j) $y = {}^{-}2$
   (k) $y > {}^{-}2$
   (l) $y < {}^{-}2$

   (m) Compare the graphs obtained in (a), (b), (c).
   (n) Compare the graphs obtained in (d), (e), (f).

2. Some ordered pairs that satisfy the condition $y = 3 + x$ are $(0,3)$, $(2,5)$, $({}^{-}3,0)$, $({}^{-}1,2)$, $({}^{-}7,{}^{-}4)$.

   (a) Plot the points associated with these ordered pairs.
   (b) Draw the line through these points.
   (c) Shade in with a colored pencil the half-plane containing the points whose coordinates satisfy the condition $y > 3 + x$.

1-6. Multiplication of Rational Numbers

In the preceding section we have sketched graphs of a few simple equations. If we wish to graph an equation like $y = {}^{-}2x$, using positive and negative numbers for x, we will need to find products like $(^-2) \cdot 3$ and $(^-2) \cdot (^-4)$. The work we have already done in this chapter and in earlier grades has prepared us for multiplication in which one or more of the factors are negative numbers. In Exercises 1-3, you used the number line to find products involving negative numbers in several cases. In Problem 12 of this section, you will be asked to find products like $(^-2) \cdot 3$ using the number line. First, however, let us consider this question from another point of view.

In earlier grades you may have considered the set of multiples of 2. If the elements of this set S are written in order, each number may be obtained from the one which precedes it by adding 2:

$$S = \{0, 2, 4, 6, 8, \cdots\}.$$

Similarly, any number (except the first) in the set of multiples of 7, when they are ordered, may be obtained by adding 7 to the number which precedes it.

$$T = \{0, 7, 14, 21, 28, \cdots\}.$$

For the set T we see that we can also say that each number may be obtained from the one which follows it by subtracting 7. For example, 28 follows 21, and $21 = 28 - 7$.

Some of you may have made multiplication tables like this:

x	3	4	5
4	12	16	20
5	15	20	25
6	18	24	30
7	21	28	35

In this table, 30 is the product of $6 \cdot 5$. This multiplication table suggests a way of thinking about products of two rational numbers when one or both of the numbers are negative.

x	$^{-}2$	$^{-}1$	0	1	2	3
$^{-}2$			0			
$^{-}1$			0			
0	0	0	0	0	0	0
1			0	1	2	3
2			0	2	4	6
3			0	3	6	9

Some of the cells in the above table have been filled from our knowledge of arithmetic. Also we have used the property that the product of a negative number and 0 is 0. Now to complete the table, let us observe, for example, as we go up in the right-hand column that each number is 3 less than the number below it. We shall refer to this column as the "3 column." Thus, the "3 column" would become

$^{-}6$
$^{-}3$
0
3
6
9

Similarly the "3 row" would become

$^{-}6$	$^{-}3$	0	3	6	9

Applying this notion to the remainder of the cells, the table would be completed as shown on the following page.

x	$^{-}2$	$^{-}1$	0	1	2	3
$^{-}2$	4	2	0	$^{-}2$	$^{-}4$	$^{-}6$
$^{-}1$	2	1	0	$^{-}1$	$^{-}2$	$^{-}3$
0	0	0	0	0	0	0
1	$^{-}2$	$^{-}1$	0	1	2	3
2	$^{-}4$	$^{-}2$	0	2	4	6
3	$^{-}6$	$^{-}3$	0	3	6	9

In particular, we see that the top row, which is the "$^{-}2$ row," is completed as shown here:

4	2	0	$^{-}2$	$^{-}4$	$^{-}6$

In other words, if we are to keep the property of multiplication of counting numbers, which we recalled earlier for multiples of 2 and 7, we must accept the following products:

$$(^{-}2) \cdot (^{-}2) = 4 \quad \text{and} \quad (^{-}2) \cdot 3 = {}^{-}6.$$

You should notice similar results in other parts of the table. For each column and each row, the difference between two consecutive numbers is a fixed amount. As far as this table is concerned, the product of two negative numbers is a positive number, and the product of a negative and a positive number (in either order) is a negative number. These conclusions are actually correct for all positive and negative rational numbers. It should be clear, however, that we have not proved this result, nor have we given it as part of a definition. We have only shown one reason why the conclusions are plausible.

Another reason for adopting the above rules for multiplication is that these rules enable us to retain the usual commutative, distributive, and associative properties. For example:

$$\begin{aligned} 3 + {}^-3 &= 0 \\ 2 \cdot (3 + {}^-3) &= 2 \cdot 0 = 0 \\ 2 \cdot (3 + {}^-3) &= 2 \cdot 3 + 2 \cdot ({}^-3) = 6 + 2 \cdot ({}^-3) \\ 6 + 2 \cdot ({}^-3) &= 0 \end{aligned}$$

From this we see that $2 \cdot ({}^-3)$ is the additive inverse of 6. Also, we know that ${}^-6$ is the additive inverse of 6.

Therefore, we must have $2 \cdot ({}^-3) = {}^-6$. In general the product of a negative and a positive number must be a negative number.

Also

$$\begin{aligned} 4 + {}^-4 &= 0 \\ {}^-5 \cdot (4 + {}^-4) &= {}^-5 \cdot 0 = 0 \\ {}^-5 \cdot (4 + {}^-4) &= ({}^-5) \cdot 4 + ({}^-5) \cdot ({}^-4) \end{aligned}$$

We have shown above that $({}^-5) \cdot 4 = 4 \cdot ({}^-5) = {}^-20$, and therefore $\quad {}^-20 + ({}^-5) \cdot ({}^-4) = 0.$

This means that $({}^-5) \cdot ({}^-4)$ is the additive inverse of ${}^-20$. We know that ${}^+20$ is the additive inverse of ${}^-20$, and therefore these two numbers must be equal and $({}^-5) \cdot ({}^-4) = {}^+20$; that is, the product of two negative numbers must be a positive number.

Exercises 1-6

1. Look at the large multiplication table which we completed in this section. In which rows do the products increase as we move to the right?
2. Using the same table, in which columns do the products decrease as we move down?
3. Give, in correct order, the products of 7 and the integers from ${}^-4$ to 6.

4. Give, in correct order, the products of $^{-}4$ and the integers from $^{-}5$ to 5.
5. Complete the following table. If it helps you see the pattern, add appropriate columns on the right, or add appropriate rows at the bottom of the table. It may also help you to first fill in the rows and columns for 0 and 1.

x	$^{-}3$	$^{-}2$	$^{-}1$	0	1	2
$^{-}5$						
$^{-}4$						
$^{-}3$						
$^{-}2$						
$^{-}1$						
0						
1						

6. Using the above table, verify the commutative property of multiplication for:
   (a) $^{-}2$ and 1 (b) $^{-}3$ and 0 (c) $^{-}2$ and $^{-}3$ (d) $^{-}1$ and $^{-}3$.
7. Illustrate the associative property of multiplication for numbers: $^{-}2$, $^{-}1$, and 5.
8. Illustrate the distributive property of multiplication over addition for the sets of numbers, using the last two numbers in each set as a sum.
   (a) $^{-}4$, 3, 8 (b) $^{-}2$, $^{-}3$, 6 (c) $^{-}10$, $^{-}8$, $^{-}1$

9. Find the products:

(a) $^-4 \cdot 0$

(b) $^-4 \cdot 2$

(c) $^-4 \cdot 5$

(d) $8 \cdot {}^-3$

(e) $17 \cdot {}^-2$

(f) $49 \cdot {}^-5$

(g) $^-6 \cdot {}^-9$

(h) $^-10 \cdot {}^-60$

(i) $^-21 \cdot {}^-43$

(j) $({}^-0.6) \cdot ({}^-1.4)$

(k) $4 \cdot 3 \cdot {}^-5$

(l) $^-6 \cdot 8 \cdot {}^-12$

(m) $^-3 \cdot {}^-2 \cdot {}^-11$

(n) $^-10 \cdot {}^-8 \cdot {}^-(\frac{2}{3})$

(o) $^-(\frac{4}{3}) \cdot {}^-(16) \cdot {}^-(\frac{3}{4})$

10. Find the products:

(a) $({}^-1) \cdot 4$ (b) $^-1 \cdot 5$ (c) $^-1 \cdot 11$ (d) $8 \cdot ({}^-1)$ (e) $77 \cdot ({}^-1)$

11. Show the use of the number line in finding the products by repeated addition:

(a) $3 \cdot {}^-2$ (b) $5 \cdot {}^-2$ (c) $4 \cdot {}^-3$.

12. State in your own words how one could use the number line to find the product $^-4 \cdot 3$. (Hint: Use the commutative property of multiplication.)

13. A football team has the ball on its own 45-yard line and then loses two yards on each of the next three successive plays.

(a) What will its new position be?

(b) Write an expression involving negative numbers to obtain the answer to (a).

14. What must n be, if $2n = {}^-18$?

15. What is n in each of the following equations?

(a) $3n = {}^-36$

(b) $5n = {}^-75$

(c) $^-2n = 10$

(d) $^-3n = 30$

(e) $^-2n = {}^-8$

(f) $^-6n = {}^-12$

16. In the following problems in multiplication put a number in the parentheses so that the statements will be correct.

(a) $(\ )\cdot 6 = {}^{-}12$

(b) $5\cdot(\ ) = {}^{-}15$

(c) $({}^{-}10)\cdot(\ ) = 100$

(d) $({}^{-}5)\cdot(\ ) = 20$

(e) $({}^{-}5)\cdot(\ ) = {}^{-}20$

(f) $11\cdot(\ ) = {}^{-}110$

(g) $({}^{-}1)\cdot(\ ) = 1$

(h) $({}^{-}7)\cdot(\ ) = 0$

(i) $1\cdot(\ ) = -1$

(j) $6\cdot(\ ) = {}^{-}36$

(k) $({}^{-}9)\cdot(\ ) = 81$

(l) $5\cdot(\ ) = {}^{-}30$

(m) $(\ )\cdot({}^{-}10) = {}^{-}90$

(n) $(\ )\cdot({}^{-}50) = 100$

(o) $({}^{-}6)\cdot(\ ) = {}^{-}60$

(p) ${}^{-}(\frac{1}{2})\cdot(\ ) = {}^{-}1$

17. Find the products:

(a) $({}^{-}6)\cdot({}^{-}10)$

(b) $({}^{-}3)\cdot({}^{-}4)$

(c) $\frac{5}{2}\cdot 6$

(d) ${}^{-}(\frac{21}{3})\cdot {}^{-}6$

(e) ${}^{-}(\frac{23}{4})\cdot {}^{-}4$

(f) $({}^{-}75)\cdot({}^{-}4)$

(g) $({}^{-}4)\cdot({}^{-}10)$

(h) $4\cdot({}^{-}10)$

(i) $({}^{-}10)\cdot 4$

(j) $({}^{-}6)\cdot({}^{-}7)$

(k) $({}^{-}15)\cdot({}^{-}4)$

(l) ${}^{-}20\cdot {}^{-}(\frac{11}{2})$

(m) $(16)\cdot({}^{-}12)$

(n) $({}^{-}16)\cdot({}^{-}12)$

(o) $({}^{-}45)\cdot({}^{-}3)$

(p) $25\cdot({}^{-}3)$

(q) $({}^{-}27)\cdot 0$

(r) $({}^{-}16)\cdot(1)$

(s) $(20)\cdot({}^{-}10)\cdot({}^{-}5)$

(t) $({}^{-}3)\cdot({}^{-}5)\cdot({}^{-}4)$

(u) $({}^{-}5)\cdot 6\cdot({}^{-}2)$

(v) $({}^{-}4)\cdot({}^{-}5)\cdot 3$

(w) $({}^{-}2)\cdot({}^{-}1)\cdot({}^{-}3)$

(x) $({}^{-}4)\cdot({}^{-}2)\cdot({}^{+}2)$

(y) $({}^{-}3)\cdot({}^{-}3)\cdot({}^{-}3)$

(z) $({}^{-}2)\cdot(2)\cdot({}^{-}2)$

1-7. Division of Rational Numbers

We know that if $3 \cdot n = 39$, then $n = 13$, since $3 \cdot 13 = 39$. Also, in the definition of rational numbers (Chapter 6, Volume I), we call $\frac{39}{3}$ (or $39 \div 3$) the rational number n for which $3 \cdot n = 39$.

$$\frac{39}{3} = 39 \div 3 = 13.$$

Let us apply the methods we have used in division of rational numbers in the seventh grade as we think of division of rational numbers involving positive and negative numbers.

Find n if $2n = {}^{-}18$.

We know $2\,({}^{-}9) = {}^{-}18$

Hence, $n = {}^{-}9$ or $\left(\frac{^{-}18}{2}\right)$

Also ${}^{-}18 \div 2 = {}^{-}9$.

In this section, we will discuss division only as the operation which is the inverse of multiplication. To find ${}^{-}8 \div {}^{-}2$, we think

$$\begin{aligned} {}^{-}8 \div {}^{-}2 = n \quad \text{or} \quad {}^{-}2 \cdot n &= {}^{-}8 \\ n &= 4, \text{ since } {}^{-}2 \cdot 4 = {}^{-}8 \\ {}^{-}8 \div {}^{-}2 &= 4. \end{aligned}$$

The question, "What is 16 divided by $^{-}4$?" is the same as the question, "By what number can $^{-}4$ be multiplied to obtain 16?" We know, ${}^{-}4 \cdot {}^{-}4 = 16$. Hence,

$$16 \div {}^{-}4 = {}^{-}4.$$

Which of the following are true statements?

(a) ${}^{-}63 \div {}^{-}9 = 7$

(b) $45 \div {}^{-}5 = 9$

(c) ${}^{-}8 \cdot {}^{-}13 = 104$

(d) ${}^{-}2 \cdot {}^{-}\left(\frac{3}{2}\right) = 3$

(e) ${}^{-}2 \div 3 = {}^{-}\left(\frac{3}{2}\right)$

(f) $3 \div {}^{-}\left(\frac{3}{2}\right) = {}^{-}2$

You should be able to show that all of these are true statements except (b) and (e).

Before starting to do the exercises, study the following and be sure that you know why they are true statements.

$$ {}^{-}7 \cdot {}^{-}5 = 35 \qquad 35 \div {}^{-}7 = {}^{-}5 \qquad 35 \div {}^{-}5 = {}^{-}7 $$

$$ {}^{-}7 \cdot 5 = {}^{-}35 \qquad {}^{-}35 \div {}^{-}7 = 5 \qquad {}^{-}35 \div 5 = {}^{-}7 $$

$$ 7 \cdot \frac{5}{7} = 5 \qquad 5 \div 7 = \frac{5}{7} \qquad 5 \div \frac{5}{7} = 7 $$

$$ 7 \cdot {}^{-}(\tfrac{5}{7}) = {}^{-}5 \qquad {}^{-}5 \div 7 = {}^{-}(\tfrac{5}{7}) \qquad {}^{-}5 \div {}^{-}(\tfrac{5}{7}) = 7 $$

What is the reciprocal of ${}^{-}(\frac{4}{3})$? We know that ${}^{-}(\frac{4}{3})$ and n are reciprocals if

$$ {}^{-}(\tfrac{4}{3}) \cdot n = 1. $$

Since $(\frac{4}{3}) \cdot (\frac{3}{4}) = 1$

and ${}^{-}(\frac{4}{3}) \cdot {}^{-}(\frac{3}{4}) = 1,$

we have $n = {}^{-}(\frac{3}{4});$

therefore, ${}^{-}(\frac{3}{4})$ is the reciprocal of ${}^{-}(\frac{4}{3})$.

Exercises 1-7

1. Find the products:

   (a) ${}^{-}4 \cdot 7$

   (b) ${}^{-}4 \cdot {}^{-}3$

   (c) $2 \cdot {}^{-}6$

   (d) $3 \cdot {}^{-}24$

   (e) ${}^{-}8 \cdot {}^{-}9$

   (f) ${}^{-}21 \cdot {}^{-}35$

   (g) ${}^{-}(\frac{3}{4}) \cdot 4$

   (h) ${}^{-}10 \cdot {}^{-}(\frac{2}{5})$

   (i) ${}^{-}(\frac{5}{6}) \cdot {}^{-}(\frac{12}{25})$

2. Find the quotients:

(a) $^-28 \div 7$

(b) $12 \div {}^-3$

(c) $^-12 \div 2$

(d) $^-72 \div 3$

(e) $72 \div {}^-9$

(f) $735 \div {}^-35$

(g) $^-3 \div 4$

(h) $4 \div {}^-(\frac{2}{5})$

(i) $\frac{2}{5} \div {}^-(\frac{5}{6})$

3. Complete the table:

x	$\frac{1}{2}x$
4	
	1
0	
$^-2$	
	$^-2$
$^-5$	
$^-6$	

4. Complete the table:

x	^-4x
2	
	$^-6$
1	
	0
	2
$^-(\frac{3}{2})$	
$\frac{3}{4}$	

5. Find r so that these sentences will be true statements.

(a) $3\ r = 17$

(b) $5\ r = {}^-10$

(c) ${}^-2\ r = 6$

(d) ${}^-3\ r = {}^-21$

(e) $2\ r = \frac{3}{8}$

(f) $3\ r = {}^-(\frac{3}{2})$

(g) ${}^-7\ r = 8$

(h) $({}^-6)r = {}^-(\frac{18}{3})$

(i) $10r = {}^-(\frac{25}{30})$

6. Write the reciprocals of each number in P:

$P = \{6,\ \frac{3}{2},\ 1,\ \frac{5}{6},\ {}^-1,\ {}^-(\frac{3}{2}),\ {}^-(\frac{7}{3})\}$

7. Divide in each of the following:

(a) $\frac{^-18}{^-9}$

(b) $\frac{^-25}{5}$

(c) $\frac{^-30}{6}$

(d) $\frac{^-30}{^-6}$

(e) $\frac{30}{^-6}$

(f) $\frac{30}{6}$

(g) $\frac{^-100}{^-20}$

(h) $\frac{^-36}{^-12}$

(i) $\frac{^-432}{12}$

(j) $\frac{^-441}{^-21}$

(k) $\frac{^-484}{22}$

(l) $\frac{^-169}{^-13}$

(m) $\frac{64}{^-16}$

(n) $\frac{750}{^-30}$

(o) $\frac{0}{^-6}$

(p) $\frac{^-39}{^-3}$

(q) $\frac{72}{^-6}$

(r) $\frac{90}{^-15}$

8. Find n if:

(a) ${}^-2 \div 3 = n$

(b) $2 \div {}^-3 = n$

9. Write ${}^-(\frac{2}{3})$ as a quotient in two ways.

10. Find n if

(a) $7\ n = {}^-6$

(b) ${}^-7\ n = 6$

11. Write two sentences, using n, in which $n = {}^-(\frac{7}{6})$ would make the sentence a true statement.

12. Find n for each of these equations.

    (a) $({}^-25)n = {}^-92$ (c) ${}^-4\,n = {}^-(\frac{4}{3})$

    (b) $({}^-92) \div ({}^-25) = n$ (d) ${}^-(\frac{4}{3}) \div ({}^-4) = n$

13. Write $\frac{92}{25}$ as a quotient in another way, using two of the numbers: 92, ${}^-92$, 25, or ${}^-25$.

14. Write two number sentences, using n, in which $n = \frac{92}{25}$ would make the sentence a true statement.

15. Complete the statements:

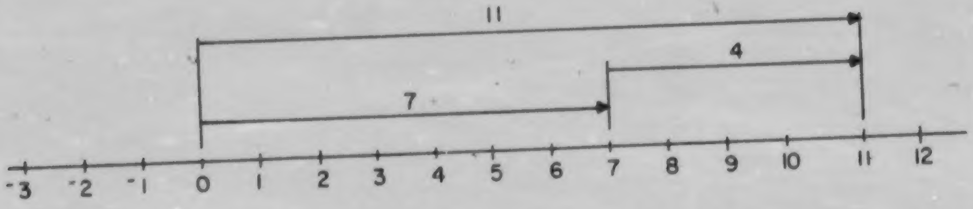
    (a) If a and b are positive or negative integers, then $\frac{a}{b}$ is the rational number x for which ______________.

    (b) If $\frac{a}{b}$ is a rational number then $\frac{a}{b}$ is positive if a and b are______________________________.

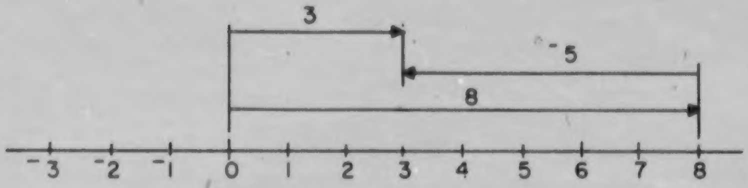
    (c) If $\frac{a}{b}$ is a rational number then $\frac{a}{b}$ is negative, if either a or b is ____________________ and the other is____________________.

1-8. Subtraction of Rational Numbers

The number line will be helpful in the understanding of subtraction of rational numbers. In arithmetic we learned, for example, that if $7 + 4 = 11$, then $11 - 4 = 7$. This statement may be explained by referring to the following figure.



We will use this property for negative numbers as well. Since $8 + ({}^-5) = 3$, then $3 - ({}^-5) = 8$. The figure may help you to understand this.



The property we are using is that whenever $a + b = c$, then $c - b = a$. To find $7 - 4$, we must find the number which, when added to 4, gives 7. Since $4 + 3 = 7$, it follows that $7 - 4 = 3$.

To find $({}^-7) - 4$, we must think of the number which, when added to 4, gives ${}^-7$. Since $({}^-11) + 4 = {}^-7$, we know that $({}^-7) - 4 = {}^-11$.

By the commutative property of addition $a + b = b + a$. Any statement about addition gives two statements about subtraction. For example:

$3 + ({}^-2) = 1$, so that $1 - ({}^-2) = 3$ and $1 - 3 = {}^-2$.

Notice also that $3 - 2 = 1$ and $3 + ({}^-2) = 1$.
So we see that: $3 - 2 = 3 + ({}^-2)$.

In other words the result of subtracting 2 is the same as adding the additive inverse of 2. The examples given above can be used to verify this general property.

Since $3 - ({}^-5) = 8$ and $3 + 5 = 8$,
then $3 - ({}^-5) = 3 + 5$.
Since $7 - 4 = 3$ and $7 + ({}^-4) = 3$,
then $7 - 4 = 7 + ({}^-4)$.
Since $({}^-7) - 4 = {}^-11$ and $({}^-7) + ({}^-4) = {}^-11$,
then $({}^-7) - 4 = ({}^-7) + ({}^-4)$.
Since $1 - ({}^-2) = 3$ and $1 + 2 = 3$,
then $1 - ({}^-2) = 1 + 2$.
Since $1 - 3 = {}^-2$ and $1 + ({}^-3) = {}^-2$,
then $1 - 3 = 1 + ({}^-3)$.

In each case we see that subtracting a number is the same as adding its additive inverse.

Exercises 1-8

1. Add the numbers in each set.

(a) $^-2,\ 5$

(b) $7,\ ^-7$

(c) $^-5,\ ^-2$

(d) $^-4,\ 1,\ 8$

(e) $^-2,\ ^-3,\ 15$

(f) $21,\ ^-6,\ ^-7$

(g) $^-8,\ ^-13,\ ^-24$

(h) $^-7,\ 50,\ ^-110$

(i) $^-23,\ ^-19,\ 14$

2. Find the sum of $^-(\frac{3}{4})$ and $(\frac{7}{2})$ and write two equations involving subtraction which can be obtained from this sum.

3. Find x in the following:

(a) $(^-5) + 2 = x$

(b) $(^-3) + x = 8$

(c) $8 + x = {}^-3$

(d) $x + (^-4) = 11$

(e) $^-(\frac{2}{3}) + {}^-(\frac{3}{2}) = x$

(f) $^-(\frac{2}{3}) + x = {}^-(\frac{13}{6})$

(g) $^-(\frac{1}{2}) + x = \frac{9}{2}$

(h) $x + \frac{4}{3} = {}^-(\frac{11}{4})$

4. Supply the missing number in each case.

(a) $8 + 5 + (\quad) = 8$

(b) $6 + (^-3) + (\quad) = 6$

(c) $(^-11) + 6 + (\quad) = {}^-11$

(d) $(^-11) + (^-6) + (\quad) = {}^-11$

(e) $(^-3) + (\quad) + (^-8) = (^-8)$

(f) $(^-3) + (^-7) + (\quad) = {}^-7$

(g) $\frac{1}{2} + (^-\frac{1}{4}) + (\quad) = \frac{1}{2}$

(h) $\frac{1}{4} + (\quad) + \frac{1}{2} = \frac{1}{4}$

5. Perform the subtractions in the following:

   (a) $(^-4) - 2$ (e) $^-(\frac{5}{4}) - (\frac{1}{4})$

   (b) $(^-6) - (^-1)$ (f) $^-(\frac{3}{2}) - (\frac{^-7}{2})$

   (c) $8 - (^-3)$ (g) $\frac{35}{3} - (^-2)$

   (d) $(^-11) - (^-13)$ (h) $\frac{75}{7} - (^-6)$

6. What are the additive inverses of

   (a) 10 (b) $^-100$ (c) $\frac{1}{2}$ (d) $\frac{7}{9}$ (e) $^-(\frac{8}{5})$ (f) $^-(\frac{49}{51})$

7. Change each part of Problem 5 to a problem of adding the additive inverse. For example, $(^-4) - 2 = (^-4) + (^-2)$.

8. Perform the following subtractions.

   (a) $(^-10) - (^-3) =$ (h) $9 - (^-3) =$
   (b) $4 - 6 =$ (i) $7 - (5) =$
   (c) $16 - 12 =$ (j) $7 - (^-5) =$
   (d) $8 - (^-2) =$ (k) $2 - 9 =$
   (e) $(^-8) - 2 =$ (l) $2 - (^-9) =$
   (f) $(^-8) - (^-2) =$ (m) $3 - 10 =$
   (g) $(^-9) - 2 =$ (n) $3 - (^-10) =$

9. Complete the following tables.

(a)

x	$2x$	$2x - 3$
$^{-}1$		
2		1
$^{-}4$	$^{-}8$	
	0	
$^{-}7$		
$^{-}9$		$^{-}21$

(b)

x	$^{-}2x$	$^{-}2x - {}^{-}1$
$^{-}1$		
0		1
	$^{-}6$	
4		
5	$^{-}10$	$^{-}9$
$^{-}2$		

Chapter 2

EQUATIONS

2-1. Writing Number Phrases

Do you like mystery stories? Have you ever imagined yourself to be a detective like Sherlock Holmes or Nancy Drew? Sometimes a mathematician must find one or more unknown numbers from certain clues. Then the mathematician works like a detective trying to solve a mystery.

For example, suppose you are trying to find a certain number. Let us call the number x. Mathematicians often use letters like "x," "y," "v," and so on to represent unknown numbers. You are given the following clue:

$$x + 5 = 7.$$

In words we may say that 7 is 5 more than the unknown number. In this example can you find the unknown number? You probably can.

Sometimes the unknown number is not so easy to find. For example, suppose you were given this problem:

> Tom bought a ticket for a football game. Altogether he paid $1.10 (or 110 cents), including the tax. If the cost of the ticket is $1.00 more than the amount of the tax, what is the amount of tax on the ticket?

Does 10 cents seem to be a reasonable guess? Recall that the total cost is 110 cents. If the tax is 10 cents, the cost of the ticket is $1.10 - $.10 or $1.00. But this is not correct, because if the cost of the ticket is $1.00, then the cost would be only $.90 more than the tax.

Let us again check the clues in the problem. You must use the clues correctly if you are to find the correct answer. To help find the amount of the tax, use the clues to write a number sentence.

Let us use x to represent the number of cents for the tax. Since the cost of the ticket is 100 cents ($1.00) more than the tax, we must add this amount to x to obtain the cost of the ticket. Thus, the cost of the ticket may be represented by $(x + 100)$. If we add the cost of the ticket $(x + 100)$, to the tax, x, we have the total cost of the ticket, $x + (x + 100)$. Thus we obtain the number sentence,

$$x + (x + 100) = 110.$$

Can you now find the amount of the tax? The correct answer is 5 cents ($.05). The correct price of the ticket is $1.05. Does $1.05 - $.05 = $1.00?

Some questions about numbers may be answered easily with a little knowledge of arithmetic. Some difficult problems, however, are more easily solved by first writing a number sentence stating the condition of the problem. In the two problems above, we used the clues to write number sentences. Each clue was a statement about numbers. Some of these numbers were known and some unknown. We then seek to find what number x satisfies the condition $x + (x + 100) = 110$.

In this chapter you will first learn to write number sentences about problems. Later in the chapter you will learn to use various properties about numbers in solving more difficult number sentences.

A sentence about numbers may be written in this form:

$$x + 7 = 9.$$

This sentence about numbers says,

> "If seven is added to a certain number x the result is nine."

The "9" is a part of the number sentence above. Another part of the sentence is "$x + 7$." These expressions, $x + 7$ and 9, are not sentences. They are only parts of sentences and are called phrases.

A phrase does not make a statement. In a sentence about numbers a phrase represents a number. A phrase that describes or represents a number is called a number phrase.

Some number phrases represent specific numbers. For example, the number phrases $(3 + 5)$, 9, $\frac{16}{2}$, III + V, and 10 represent specific numbers. In each of these examples, the value of the number phrase is known, or it can be determined. A number phrase which represents a specific number is called a closed number phrase, or more simply a closed phrase.

What number is represented by $x - 4$? We cannot determine the number unless we know the value of x. Thus, $x - 4$ may have many different values. Number phrases which do not represent a specific number are called open number phrases, or more simply open phrases. We may think of an open phrase as one whose value is "open" to many possibilities. Examples of open phrases are, $(x - 4)$, $7y$, $(2 + z)$, $\frac{B}{4}$, and $(3x + 2x)$.

To solve problems by using number sentences you must be able to translate the clues given in the problem into an open sentence. To do this you must express the numbers in the problem as number phrases. Earlier in this section, we used the number sentence,

$$x + 5 = 7.$$

Is the value of 7 known? What about $x + 5$? Is 7 an open phrase? What about $x + 5$?

To work with number phrases you must also be able to translate the phrase into words. The phrase $x + 5$ may be translated as "the number x increased by five." Can $7x$ be translated as "seven times the number x"?

Sometimes pupils are confused because an open phrase such as $x + 7$ may have many different translations. For example, other translations are:

"The number seven added to x,"
or "the number x increased by seven,
or "the sum of x and seven,"
or "seven more than the number x."

However, all of the translations have the same mathematical meaning. Furthermore, all of the English translations mean the same as "$x + 7$." With practice you will learn to understand the different ways of expressing a number phrase.

Class Exercises 2-1

1. Translate each of the following number phrases:

   (a) The sum of x and 5

   (b) The number x decreased by 3

   (c) The product of 8 and x

   (d) One fourth of the number x

   (e) The number x increased by 10

   (f) The number 7 multiplied by x

   (g) The number which is 11 subtracted from x

   (h) The number x divided by 2

   (i) The number which is 6 less than x

   (j) The number x decreased by 9

2. For each one of the number phrases in Problem 1, find the number represented by the phrase if $x = 12$ in each part.

3. Translate each of the following number phrases into words:

   (a) $x + 1$ (d) $\frac{18}{x}$

   (b) $x - 3$ (e) $4x$

   (c) $2x$ (f) $^{-}6 + x$

4. Find the number represented by each of the number phrases in Problem 3 if $x = 6$.

5. Find the number represented by each of the number phrases in Problem 3 if $x = {}^{-}2$.

6. Assume that during January you saved d dollars. In February you saved 5 dollars more than you saved in January.

   (a) Write an open phrase which represents the number of dollars you saved in February.

   (b) Write an open phrase which represents the total number of dollars you have saved.

7. Write open phrases representing each of the following:
   (a) The number of cents in d dimes
   (b) The number of gallons in q quarts
   (c) The number of feet in y yards
   (d) The number of cents in one more than n nickels
   (e) The number of inches in three less than f feet

8. Write open phrases representing each of the following:
   (a) A number plus four
   (b) The sum of a number and twice the number
   (c) A number increased by seven
   (d) Five subtracted from a number
   (e) A number subtracted from five
   (f) The product of nine and a number
   (g) The quotient of a number divided by ten
   (h) The quotient of ten divided by a number
   (i) A number subtracted from twice the number
   (j) Three times a number divided by two times the number

Exercises 2-1

1. The unknown number is not always represented as x. Translate each of the following number phrases into symbols using the letter of each part as the unknown number. For example, in Part (a) use "a" as the unknown number.
   (a) The sum of six and a number
   (b) Eight times a number
   (c) Eight times a number and that amount increased by 1
   (d) Three subtracted from eight times a number

(e) The amount represented by eight times a number divided by 4

(f) Two times a number and that amount increased by 3

(g) Five multiplied by the sum of a number and 2

(h) Ten less than seven times a number

(i) Twelve divided by the sum of a number and 1

(j) The product of two factors, one of which is the sum of 3 and a certain number and the other of which is the sum of 4 and the same number

2. Find the number represented by each of the number phrases in Problem 1 if the unknown number is $^{-}3$.

3. Translate each of the following open number phrases into words. Write the word "number" to represent the unknown number in each phrase.

   Example: $y + 3$ A number increased by three.

   (a) $2n + 5$

   (b) $6 - 3q$

   (c) $7(b - 1)$

   (d) $\frac{5 - d}{2}$

   (e) $15 + 2w$

4. Find the number represented by the open phrase $2n + 5$ for each of the following values:

   (a) $n = 5$

   (b) $n = {^{-}5}$

   (c) $n = 0$

   (d) $n = {^{-}1}$

5. Find the number represented by the open phrase $6 - 3q$ for each of the following values:

   (a) $q = 0$

   (b) $q = {^{-}1}$

   (c) $q = {^{+}1}$

   (d) $q = 5$

6. Write open number phrases to represent each of the following:

   (a) Ann's age if Ann is three years older than her brother, and he is x years old

   (b) The cost of ten pencils at y cents per pencil

   (c) The number of cents in q quarters

   (d) A boy's age five years from now if he is y years old now

   (e) A girl's age six years ago if she is z years old now

   (f) The total number of points Bob scores in two basketball games, if he scored g points in the first game and twice that number in the second game

   (g) The number of dollars Mary has after spending 6 dollars, if she had n dollars at first

   (h) The sum of the three ages, if Kathy's father's age is four times her age, and her mother's age is twenty more than Kathy's age

   (i) The total number of cents Mike has, if he has k nickels and the number of dimes he has exceeds the number of nickels by 1

7. The closed number phrases, $1 + 2$, $2 + 3$, $3 + 4$, and $4 + 5$, represent the sums of pairs of consecutive numbers. Write an open phrase representing the sum of any two consecutive numbers. Hint: if n represents the smaller of the two consecutive numbers, can the larger number be represented by $(n + 1)$?

8. Write an open phrase representing the sum of any two consecutive odd numbers.

9. Write an open phrase representing the sum of any three consecutive even numbers.

10. Write an open phrase representing any two consecutive multiples of ten.

2-2. Writing Number Sentences

All of us use sentences every day. We use sentences in talking. When we read, we read sentences. In mathematics we need to deal with many kinds of sentences. We use sentences to explain mathematics and to discuss mathematics. Some mathematical sentences make statements about numbers, and it is this particular kind of sentence which we are studying in this chapter. Consider the following sentences:

> "The sum of 8 and 7 is 15."
> "$x + 3 = 8$."
> "Five is greater than the sum of one and two."
> "$3 < 2 + 4$."
> "The sum of 3 and 4 is not equal to the product of 3 and 4."

Each of these sentences consists of two number phrases connected by a verb or verb phrase. In number sentences verbs or verb phrases are represented by the symbols "$=$," "$<$," "$>$," and "$\neq$."

A number sentence using the symbol "$=$" indicates equality. The sentence, $x + 3 = 8$, makes the statement that "$x + 3$" and "8" are different names for the same number. We call such a sentence an equation. When $x = 5$, the statement, $x + 3 = 8$, is true and when x is not 5, the statement is false.

Consider the sentence, "$x - 4 > 7$." Is this sentence an equation? What does the symbol "$>$" represent? Does it have the same meaning as "$=$"? This sentence indicates that the number represented by "$x - 4$" is greater than the number represented by "7." Such a sentence is called an inequality. Inequality means "not equal." Other examples of inequalities are "$x + 1 < 15$" and "$2x \neq 18$."

Some sentences are true. For example, "$4 + 5 = 3 \cdot 3$" and "The sun sets in the west," are true sentences. Sentences may not be true, however. "$3 > 2 + 4$" and "Abraham Lincoln was the first president of the United States" are not true sentences.

Consider the sentences,

"Jimmy was at Camp Holly all day yesterday,"
and "$x + 3 = 8$."

Are they true? Are they false? You may answer, "I don't know. Which Jimmy do you mean? To what number does "x" refer?" These sentences are neither true nor false, because they contain words or symbols which do not refer to only one thing. "Jimmy" can mean any boy with that name, and "x" can stand for any number. You might look at the camp records and say that, if "Jimmy" means Jimmy Mills of Denver, the first sentence is true; but, if it means Jimmy Shultz of Cincinnati, then it is false. The second sentence is true if $x = 5$, but it is false if $x = 6$ or if x is any number other than 5.

What can you say about the truth of the three following sentences?

"$13 - x = 7$."
"George was the first president of the United States."
"$3 + x = x + 3$."

These sentences are similar in that each contains a word or symbol which can refer to any one of many objects. Do you see any difference between the first two sentences and the third? Can the first two sentences be true? Can the first two sentences be false? Can the last sentence ever be false?

Suppose a number sentence involves a symbol like "x" or "y." If the symbol can refer to any one of many numbers the sentence is called an open sentence. It is not necessarily a true sentence. It is not necessarily a false sentence. It leaves the matter open for further consideration.

Look at this equation:

$$x + 7 = \frac{10}{x - 2}$$

This equation is composed of three parts: a verb "=," and two open phrases, "$x + 7$" and "$\frac{10}{x - 2}$." The equation states that for a certain number x these two open phrases represent the same number. Can you discover such a number x? Can you find more than one? Try some numbers. After working for a while you might say, "The sentence is true if $x = 3$ or $x = {}^{-}8$, but it is false if x is any other number." The numbers 3 and ${}^{-}8$ are called solutions of the open sentence. The set $\{3, {}^{-}8\}$ is called the set of solutions of the open sentence.

When we find the entire set of solutions of an open sentence, we say that we have solved the sentence. An equation is a particular kind of number sentence which involves the verb "=." To solve an equation means to find its entire set of solutions. The set of solutions of an equation may contain one member or it may contain several members. It might even be the empty set.

You may have already used a special kind of equation. For example, to find the number of square units of area in a rectangle you used the following:

$$A = lw$$

This is an abbreviation of a rule. In words, this rule is:

> "The number of square units of area in a rectangle is (or, is equal to) the product of the number of units in the length and the number of like units in the width."

When such a rule is abbreviated and written in the form of an equation it is called a formula. If the length and width of a rectangle are known, then this formula may be used to find the area of that rectangle.

Can you determine the set of solutions for the inequality $x - 4 > 7$? How large must the number x be in order for the inequality to be true? Is $5 - 4 > 7$? Is $7 - 4 > 7$? Is $12 - 4 > 7$? Do you see that "$x - 4 > 7$" is true if x is any number greater than 11? Also, "$x - 4 > 7$" is false for any other value of x. Thus the set of solutions of this inequality is the set of all numbers which are greater than 11.

Class Exercises 2-2a

In Problems 1-4 below use your knowledge of arithmetic to find the solution set for each of the number sentences. Remember that each member of the solution set must, when it is used as a replacement for the unknown, result in a true sentence.

1. (a) $x + 3 = 5$
   (b) $y + 3 > 5$
   (c) $k + 13 = {}^{-}15$
   (d) $m + 25 = 31$
   (e) $s + 25 < 31$
   (f) $t + 10 \neq 5$

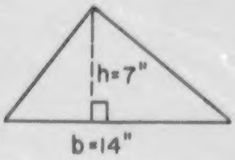
2. (a) $x + ({}^{-}7) = 2$
   (b) $y + ({}^{-}7) > 2$
   (c) $n + ({}^{-}9) = {}^{-}2$
   (d) $x + ({}^{-}3) = 6$
   (e) $p + ({}^{-}15) = {}^{-}1$
   (f) $x + ({}^{-}15) < {}^{-}1$

3. (a) $4b = 12$
   (b) $4a \neq 12$
   (c) $5w = 35$
   (d) $5m < 35$
   (e) $13x = {}^{-}13$
   (f) $7y = {}^{-}56$

4. (a) $\frac{n}{3} = 2$
   (b) $\frac{a}{3} < 2$
   (c) $\frac{k}{8} = {}^{-}2$
   (d) $\frac{d}{9} > 2$
   (e) $\frac{h}{{}^{-}3} = 5$
   (f) $\frac{s}{{}^{-}3} = {}^{-}7$

5. A formula for finding the perimeter of a rectangle is $p = 2\ell + 2w$. Find the perimeter of a rectangle whose length is 7 feet and whose width is 4 feet.

6. Use the formula $a = \frac{1}{2}bh$ to find the number of square units of area in the triangle shown at the right.



Equations are used in many ways in many different fields. We solve equations to find the currents in an electrical network when we know the voltages and the resistances, to design airplanes or space ships, to find out what is happening in a cancer cell.

We also use equations to predict the weather. We now know methods for predicting tomorrow's weather very accurately. The only trouble is that these methods require the solution of about a thousand equations with the same number of unknowns. Even with the best of the modern high speed computers, it would take two weeks to compute the prediction of tomorrow's weather. Therefore, the meteorologists (look up this word) make many approximations. They simplify the equations in such a way that they can compute the prediction in a short enough time. They will be able to make better predictions when we know more efficient ways to solve many equations with many unknowns.

Many leading mathematicians are working on such problems and continue to seek new methods for solving equations. In universities, government and industrial laboratories there are actually thousands of mathematicians who are working every day at solving equations. Some use big new computing machines. Others use the kind of method you are now learning and work with pencil and paper as you do.

Exercises 2-2a

1. Translate each of the following number sentences into symbols.

   (a) The number x increased by 5 is equal to 13.

   (b) The number 3 subtracted from x is equal to 7.

   (c) The product of 8 and x is equal to 24.

   (d) When x is divided by 4 the quotient is 9.

   (e) Ten more than the number x is 21.

   (f) The number x multiplied by 7 is equal to $^{-}35$.

   (g) The number 11 subtracted from x is $^{-}5$.

   (h) The number 6 less than x is 15.

   (i) The number x divided by 2 is equal to $^{-}7$.

2. For each of the equations you wrote in Problem 1, find the set of solutions by using your knowledge of arithmetic.

3. Translate each of the following number sentences into symbols.

   (a) The number x increased by 2 is greater than 4.

   (b) The number x multiplied by 5 is less than 10.

   (c) The result of dividing x by 7 is greater than 2.

   (d) Three less than the number x is greater than 6.

   (e) The number x decreased by 5 is less than 13.

   (f) The product of 3 and the number x is greater than $^-9$.

4. For each of the inequalities you wrote in Problem 2, use your knowledge of arithmetic to find the set of solutions.

5. Translate each of the following number sentences into words. Use the term "a number" or "a certain number" to represent the unknown number.

   (a) $y + 2 = 5$

   (b) $z + (^-3) = 7$

   (c) $2a = {^-10}$

   (d) $h + (^-5) < 9$

   (e) $5m < 15$

   (f) $7 + (^-k) = 2$

   (g) $d + (^-3) < 4$

   (h) $\frac{w}{3} > 9$

   (i) $k + (^-7) = {^-2}$

   (j) $\frac{c}{^-30} = 6$

6. Using your knowledge of arithmetic, find the set of solutions for each of the number sentences in Problem 5.

7. What is the area of a square whose length is 15 inches? Use $A = s^2$ as the formula for the area.

8. A formula used in finding simple interest is written $i = prt$, where

   i is the interest in dollars,
   p is the principal (or amount borrowed),
   r is the rate (or per cent) of interest per year,
   t is the time in years.

Find the interest for a bank loan of \$750 for 3 years at 6% interest.

9. To find the circumference of a circle, the formula $c = 2\pi r$ may be used, where r stands for the radius. Find the circumference of a circle whose radius is 10 inches. (Use $\frac{22}{7}$ or 3.14 as an approximation to π.)

10. The formula: $d = rt$ may be used to find the total distance traveled if the rate of travel does not vary, where d is the measure of the distance, t is the measure of the time, and r is the rate. For example if d is measured in miles and t in hours, r will be in miles per hour. Find the distance traveled by an automobile moving at a rate of 45 miles per hour (m.p.h.) for 13 hours.

*11. To find 19% of \$750 you may use the percentage formula,

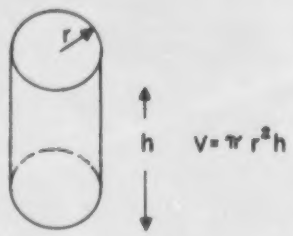
$$p = rb$$

where p is the percentage; r is the rate (or percent); and b is the base. In this problem, $r = 19\%$ or 0.19 and $b = \$750$. Find the value of p for this problem.

*12. Find the area of the floor of a circular room whose radius is 13 feet. The formula is $A = \pi r^2$. (Use $\frac{22}{7}$ or 3.14 for π).

*13. (a) Find the volume of the cylindrical tank pictured at the right if the radius is 1 foot and the height is $3\frac{1}{2}$ feet. (Use $\frac{22}{7}$ for π.)

(b) Find the capacity of the tank in gallons. One cubic foot holds $7\frac{1}{2}$ gallons.



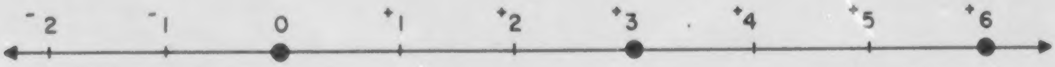
*14. The formula $F = \frac{9}{5}C + 32$ may be used to convert a temperature reading on a Centigrade thermometer to a temperature reading on a Fahrenheit thermometer. Find the correct Fahrenheit temperature reading for each of the following readings on a

Centigrade thermometer.

(a) 0° (b) 100° (c) 37°

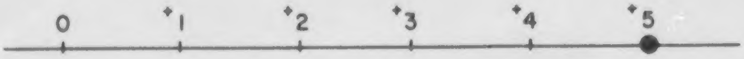
Graphs of Solution Sets

We can draw a picture to represent a set of numbers by associating the numbers with points on a line. Consider the set $\{0, 3, 6\}$. Each element is a number associated with a point on the number line. We draw a picture to represent this particular set by marking a heavy dot on the number line as shown below:

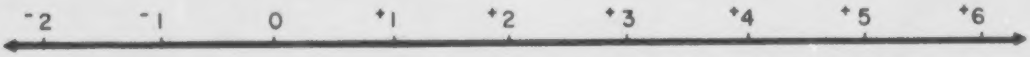


This drawing is called a graph of the set $\{0, 3, 6\}$. It is sometimes useful to draw graphs of solution sets of open sentences.

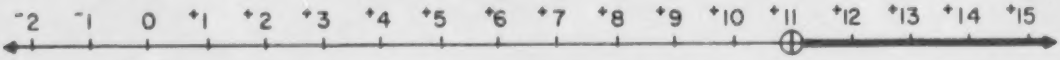
The open sentence, "$x + 3 = 8$," has the solution set $\{5\}$. The graph of this set shows a heavy dot only on the point which corresponds with 5.



What is the solution set of "$x + 3 = 3 + x$"? This sentence is true for any number we choose as a replacement for x. The solution set for this equation is the solution set of all numbers. The graph of this set is made by drawing a heavy shaded line along the entire number line as shown below.



Consider the inequality $x - 4 > 7$. The set of solutions for this inequality is the set of all numbers which are greater than 11. This set of solutions is represented on the number line as shown below:

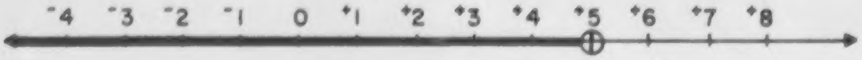


The number "11" is not in the set. We indicate this by drawing a circle around the point corresponding to 11 on the number line. The part of the number line to the right of the 11 is shaded showing that all points to the right of 11 are in the set of solutions.

What is the solution set for the inequality shown below?

$$x - 4 < 1$$

You should find that the set of solutions is the set of all numbers less than $^{+}5$. On the number line this is represented as a circle around the point corresponding to $^{+}5$ and a heavy shaded line drawn along all points of the number line which lie to the left of $^{+}5$. Here is the drawing:

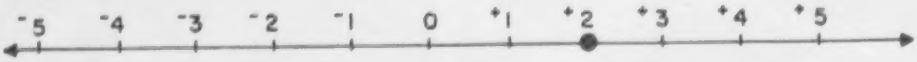


We have found that the sets of solutions for different number sentences may be different. Some of the solution sets contain only one member. Such sets may be represented by a single large dot on a drawing of the number line. The dot is drawn at the point which corresponds to the number in the set of solutions. If the set of solutions is the set of all numbers, we may draw a heavy, shaded line along the entire number line. In this case, the solution set is represented by the entire number line. The sets of solutions for inequalities are represented by a part of the number line. All of the inequalities we discussed were represented by half-lines on the number line. A circle was used to indicate a point not included in the set of solutions.

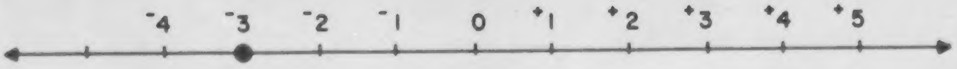
Class Exercises 2-2b

What is the solution set pictured in each of the following graphs?

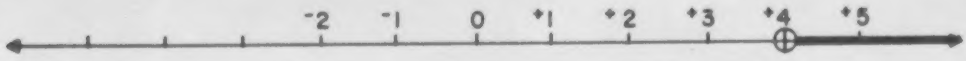
1.



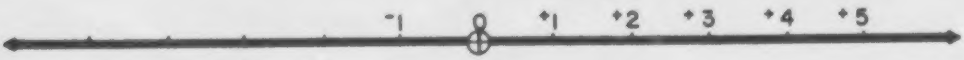
2.


-4
-3
-2
-1
0
+1
+2
+3
+4
+5

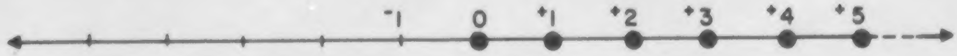
3.



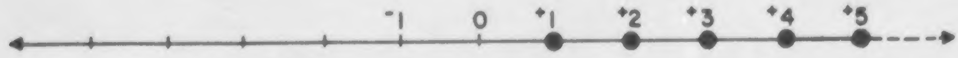
4.



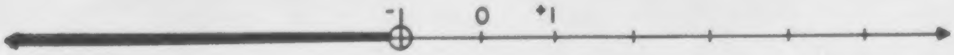
5.



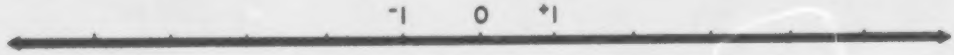
6.



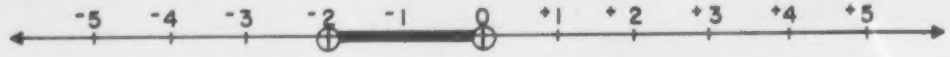
7.



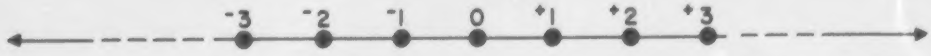
8.



9.


-5
-4
-3
-2
-1
0
+1
+2
+3
+4
+5

10.



You are not yet ready to solve very complicated equations or inequalities. For example, it is much more difficult to find the set of solutions for this number sentence,

$$x^2 < 9$$

than it was to find the set of solutions for the equations and inequalities we discussed earlier. Other number sentences may be even more complicated. You will learn much more about these complicated number sentences later in this chapter and again when you study algebra next year.

Exercises 2-2b

1. Using your knowledge of arithmetic, find the set of solutions for each of the following number sentences.

   (a) $x + 2 = 6$

   (b) $4 + x = 0$

   (c) $2x = 6$

   (d) $3x < 3$

   (e) $x + {}^{-}4 > 1$

   (f) $\frac{x}{5} = {}^{-}1$

   (g) $2x < 10$

   (h) ${}^{-}4 - x > 1$

2. For each of the number sentences in Problem 1 represent the set of solutions on a number line.

3. Using your knowledge of arithmetic, find the set of solutions for each of the following number sentences.

   (a) $x + 1 = 1 + x$

   (b) $y + {}^{-}1 > 0$

   (c) $1 - b > 0$

   (d) $a + 2 = 1 + a$

   (e) $3w = {}^{-}15$

   (f) $14 + x = 13$

   (g) $13 - x = 14$

   (h) $\frac{2}{x} = {}^{-}1$

4. For each of the number sentences in Problem 3 show the set of solutions on a number line.

5. Sometimes an equation or an inequality is only part of a sentence. Just as you can build longer sentences out of shorter ones by using such words as "and," "or," and "but," you can join number sentences together to make longer ones. Such sentences are called compound sentences.

   Consider the compound number sentence

   "$x + {}^{-}4 < 7$ and $x + {}^{-}1 > 0$."

   In order to be a solution of this sentence, a number x must be a solution of both the sentence "$x + {}^{-}4 < 7$" and the sentence "$x + {}^{-}1 > 0$."

   The elements of the solution set of the sentence are the numbers which are in both the solution set of "$x + {}^{-}4 < 7$"

and the solution set of "$x + {}^{-}1 > 0$."

The set of solutions of "$x + {}^{-}4 < 7$" is the set of all numbers less than 11.

The set of solutions of "$x + {}^{-}1 > 0$" is the set of all numbers greater than 1.

What is the set of solutions of the compound sentence? Show this set on a number line.

6. Is the set of solutions for the compound sentence in Problem 5 the intersection of the sets of solutions for the two inequalities or is it the union of the two sets of solutions?

7. For each of the following compound sentences find the set of solutions.

   (a) $x + {}^{-}2 < 7$ and $x + 4 > 6$

   (b) $x + {}^{-}3 = 6$ and $x + {}^{-}3 > 6$

   (c) $2x > 6$ and $\frac{x}{2} < 3$

8. For each of the compound sentences in Problem 7 represent the set of solutions on the number line.

*9. (a) Find the set of solutions for $x^2 = 9$. (There are two possible solutions.)

   (b) Represent the set of solutions for Part (a) on the number line.

*10. (a) Find the set of solutions for $x^2 < 9$.

   (b) Represent the set of solutions for Part (a) on the number line.

*11. (a) Find the set of solutions for the following compound sentence:

$$x + 7 = 6 \text{ or } 2x + {}^{-}1 = 5.$$

   (Note: In mathematics "or" means either the first or the second or both the first and the second.)

(b) Represent the set of solutions for Part (a) on the number line.

*12. (a) Find the set of solutions for $x + {}^-1 = 4$ or $x + {}^-1 > 4$. (This compound sentence is sometimes abbreviated $x + {}^-1 \geq 4$.)

(b) Show the set of solutions for Part (a) on the number line.

*13. Find the set of solutions for $x < 10$ or $x + {}^-9 > 0$.

In solving problems, the translation of number sentences from the English language, or words, into mathematical language, or symbols, is often the most important part of the task. Also it is often the most difficult part of the task. When you are faced with problems such as those above, think about them carefully. Often you will find that the problem is only asking, in a complicated way, for the solutions of a number sentence. By writing the sentence in symbols the set of solutions may be easier to find.

It is important that you understand that the open sentences you write are always about numbers. The problems may refer to inches or pounds or years or dollars, but your open sentences must be about numbers only.

Consider the problem, "The sum of a certain number and eight is equal to two more than the product of four and the number. What is the number?" The problem asks us to find a certain number. We represent the number with a letter such as x. We must define the meaning of x. That is, we must state what it is that the letter x represents. This is the first step in writing an open sentence, and it is a very important part of the work done in finding the solution. In this case the meaning of x is obvious: "Let x represent the unknown number." (In more difficult problems the definition of the letter used may be more complex.)

The next step is to describe the numbers in the sentence, using the symbol for the unknown number where necessary in writing phrases. The phrases for this problem are:

"the sum of the number and eight" - - - - - $\underline{x + 8}$

"two more than the product of
four and the number" - - - - - - - - - - - - - $\underline{4x + 2}$

We now have two open phrases. For other problems there may be more than two phrases. Using these phrases we write the open sentence:

$$x + 8 = 4x + 2.$$

Is this sentence an equation or an inequality? Can you guess what number will make the sentence true? The solution is 2.

Many of the problems in this section may be solved easily by using arithmetic, and you may question why it is necessary to write open sentences expressing the condition of such easy problems. Remember that in this section we are not concerned primarily with the solutions. You should concentrate on the writing of the open sentences. Later in this chapter you will learn to use properties of numbers in finding the set of solutions for more difficult problems.

Class Exercises 2-2c

For each of the following problems write an open sentence stating the condition of the problem.

1. A train travels at 80 miles per hour. How long does it take for this train to make a 560 mile trip?

   First: Let "t" represent the number of hours the train travels.

   (Note that we specify that the "t" represents a number. In this case it represents the number of hours the train travels. It is not sufficient to say, "Let t represent the time.")

   Second: Write a phrase representing the number of miles the train travels in t hours.

   Third: Write an equation stating the condition of the problem.

2. Mary is fourteen years old. She is five years older than her brother. How old is the brother?

3. A boy is four years younger than his sister. If the boy is ten years old, how old is the sister?

4. A boy bought a number of model plane kits costing 25 cents each. He spent 75 cents. How many kits did he buy?

5. A boy's age seven years from now will be 20. How old is the boy now?

6. How many feet are there in a board having a length of 72 inches?

7. How many feet are there in a board 5 yards long?

8. Ann was 3 years old ten years ago. How old is Ann at the present time?

9. How many dollars may be obtained in exchange for a total of 450 pennies?

10. If three dollars is added to twice the money Dick has, the result is less than twenty-three dollars. How much money does Dick have? (Will your open sentence for the condition in this problem be an equation or an inequality?)

11. At a certain speed a plane will travel more than 500 miles in two hours. For what speeds is this true?

12. If one is added to twice a girl's age the result is nineteen. What is the girl's age?

13. A man drove a total distance of 240 miles at an average speed of 40 miles per hour. How long did it take for the drive?

14. If a baby sitter earns 65 cents per hour, how much will she earn in 5 hours?

Exercises 2-2c

For Problems 1, 2, and 3, write open sentences for each condition stated.

1. A boy is 7 years older than his sister.

   Let x represent the number of years in the boy's age.
   The sister's age is represented as $x - 7$.

   Write equations showing that:

   (a) The sum of their ages is 21.

   (b) The boy's age is equal to two times the sister's age.

   (c) Three times the sister's age equals seven more than the boy's age.

   (d) The product of 2 and the boy's age equals the product of 4 and the girl's age.

2. Joan has twice as much money as does Cathy.

   Let m represent the number of dollars that Cathy has; then, $2m$ represents the number of dollars Joan has.

   Write equations showing that:

   (a) Together they have \$15.

   (b) If Joan spends \$5, she and Cathy will have the same amount of money.

   (c) If Cathy spends three dollars, Joan will have five times as much money as does Cathy.

3. The length of a rectangle is four feet more than the width.

   Let w be the number of feet in the width.
   Then $w + 4$ is the number of feet in the length.

   Write equations showing that:

   (a) The width when doubled is the same as the length increased by three.

(b) Assume the length is doubled. To have the same measure, the product of 3 and the width must be increased by 1.

(c) Twice the width added to twice the length is equal to 36. (This is the perimeter of the rectangle.)

For Problems 4-14 write open sentences expressing the condition of the problem.

4. In ten more years Mr. Smith will be forty years old. How old is he now?

5. If Bill earns five dollars more, he will have earned a total of twelve dollars. How much has he already earned?

6. A girl is two times as tall as her brother. If the girl is 64 inches tall, how tall is her brother?

7. Paul was 14 years old in 1958. In what year was he born?

8. Twenty per cent of a number is 10. What is the number? (Hint: $20\% = \frac{1}{5}$.)

9. A carpenter saws a 50-inch board into two pieces. One piece is 10 inches longer than the other piece. Find the length of the shorter piece.

10. In a class election Marge received 5 votes more than Bruce. How many votes did Bruce receive if there were 35 voters in the class? (Assume that each voter voted for either Marge or Bruce but not for both.)

11. If a number is added to twice the number, the sum is less than 27. For what numbers is this true?

12. The population of Minneapolis and St. Paul combined is more than twice the population of St. Paul alone. What is the population of St. Paul if the combined population is one million?

13. Pat and Mike are twins but they do not weigh the same amount. If Pat weight 105 pounds, how heavy is Mike?

14. Last year a boy earned more than one hundred twenty dollars on his paper route. What were his average monthly earnings?

*15. Mr. Smith is 4 times as old as his son. In 16 years he will be only twice as old. What are their ages now? (Hint: If the son is x years old now, how old will he be in another 16 years?)

*16. In a particular triangle one angle is twice as large as another. The third angle is three times as large as the smaller of the two other angles. How many degrees are there in the measure of each angle? (Hint: How many degrees are there in the sum of the measures of the angles of a triangle?)

*17. The sum of three consecutive whole numbers is 123. What are the three numbers?

*18. Bob has $1.25 in nickels and dimes. He has three times as many nickels as dimes. Find how many of each he has.

2-3. Finding Solution Sets

We know that there are many different ways to express any number. For instance $25 = \frac{50}{2} = \frac{75}{3} = 30 - 5 = 5^2 = (31 - 6)$. Of all these ways of expressing the number twenty-five, 25 is the simplest. Consider the following equations:

$$x + 3 = 8, \quad x + 1 = 6, \quad 2x = 10, \quad x + (^-2) = 3, \quad 10 = 2x, \quad 8 = 3 + x.$$

Each of these equations has the solution $x = 5$; that is, if in each equation we replace x by 5 we have a true sentence and if we replace x by any number different from 5, the sentence is false. These equations are called equivalent because all the solution sets are the same. In fact, the equation $x = 5$ could also be included in the list. Just as 25 is the simplest way to express the various numbers indicated at the beginning of this section, so $x = 5$ is the simplest equation equivalent to the list of equations given above.

Class Exercises 2-3a

1. Find six equations equivalent to the equation $x + 1 = 4$.
2. Find six equations equivalent to the equation $x = 3$.
3. Find six equations equivalent to the equation $2x = 12$.
4. What methods can you discover to get from a given equation one or more equations equivalent to it?
5. If one equation is equivalent to a second equation, is the second equivalent to the first? Why?

*6. Are the following two equations equivalent: $x = x + 3$, $x = x - 1$?

Consider, for instance, the equation

$$x + 3 = 7.$$

If x is replaced by a solution of this equation, $x + 3$ and 7 are two different names for the same number. Hence, for instance, $(x + 3) + 4$ and $7 + 4$ must be names for the same number, whenever x is replaced by a solution of the given equation. Since it is probably easier to see this in terms of numbers first, consider this:

$$10 = (15 - 5).$$

Hence,

$$10 + 5 = (15 - 5) + 5.$$

We have added the same number to a given number, 10, expressed in two ways. These are examples of the

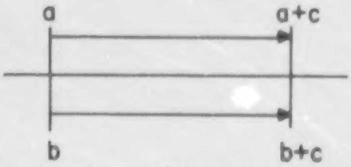
> Addition Property of Equality. If two numbers, a and b, are equal (that is, if a and b are two different names for the same number), then if you add the same number to each of them, the two sums will be equal.

That is,

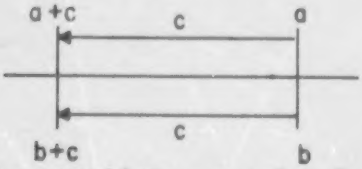
$$\text{if } a = b, \text{ then } a + c = b + c.$$

Graphically, we would have the following figures.

If $c > 0$



If $c < 0$



To see how this applies to the equation: $x + 3 = 7$, notice that if x denotes a solution of this equation, $x + 3$ and 7 are two names for the same number. Hence in the additive property,

$x + 3$ plays the role of the letter a,
7 plays the role of the letter b,

if x is a solution of the equation. Hence, if x is a solution of

$$x + 3 = 7$$

it is also a solution of

$$(x + 3) + c = 7 + c,$$

no matter what number c is. A brief way of describing this is to say that we get the second equation from the first by "adding the same number, c, to both sides of the equation."

The most useful choice of c in the above is $^{-}3$, for

$$(x + 3) + {}^{-}3 = 7 + {}^{-}3.$$

Then, by the associative property for addition, this is equivalent to

$$x + (3 + {}^{-}3) = 7 + {}^{-}3.$$

Hence

$$x + 0 = 4,$$
$$x = 4.$$

Thus, by the addition property of equality, if any number is a solution of $x + 3 = 7$, it is a solution of $x = 4$. Conversely, if any number is a solution of $x = 4$, we may add 3 to both sides of the equation to get

$$x + 3 = 7$$

and see that if any number is a solution of $x = 4$, it is also a solution of $x + 3 = 7$. Hence the equations:

$$x = 4 \quad \text{and} \quad x + 3 = 7$$

are equivalent equations; that is, their solution sets are the same.

By this means we can show that:

$$a = b \quad \text{and} \quad a + c = b + c$$

are equivalent equations; that is, if $a = b$, then

$$a + c = b + c, \text{ and if } a + c = b + c, \text{ then } a = b.$$

Using this result it also follows that $x = 3$ and $x + 7 = 10$ are equivalent equations since we got the second equation from the first by adding 7 to both sides of the equation $x = 3$.

Class Exercises 2-3b

1. Find the solution set of each of the following equations and check your solutions:

   (a) $x + 4 = 10$

   (b) $x + {}^{-}2 = 5$

   (c) $3 = x + 4$

   (d) ${}^{-}2 = x + {}^{-}3$

2. Use the distributive property to simplify each of the following:

   (a) $2x + 3x = ?$

   (b) $x + 5x = ?$

   (c) $({}^{-}2)x + 3x = ?$

   (d) ${}^{-}x + 7x = ?$

   (e) ${}^{-}x + ({}^{-}2)x = ?$

   (f) $({}^{-}2)x + ({}^{-}5)x = ?$

Suppose instead of adding a known number to both sides of the equation $x + 7 = 10$ we add an unknown, $2x$. Then we would have

$$2x + (x + 7) = 2x + 10.$$

We can write the left side, using the associative property, as

$$(2x + x) + 7.$$

Now, $x = 1 \cdot x$ and hence $2x + x = 2x + 1 \cdot x = (2 + 1)x = 3x$, using the distributive property. Thus, if we add $2x$ to both sides of the equation $x + 7 = 10$, we have

$$3x + 7 = 2x + 10.$$

This is all very well, but suppose we were given the last equation. How could we get back to the first equation? Since we got the last equation by adding $2x$ to both sides of the first, we should be able to get the first equation by adding $^-(2x)$ to both sides of the last. Let us see if this is so. Then

$$^-(2x) + 3x + 7 = {}^-(2x) + 2x + 10.$$

Now $^-(2x) + 2x = 0$ since $^-(2x)$ is the additive inverse of $2x$. But what is $^-(2x) + 3x$ equal to? We find it this way:

$$^-(2x) + 3x = (^-2)x + 3x = (^-2 + 3)x = 1 \cdot x = x,$$

using the distributive property. Thus, by adding $^-(2x)$ to both sides of the equation $3x + 7 = 2x + 10$ we get $x + 7 = 10$ as an equivalent equation. Since the solution of this equation is 3, the solution of $3x + 7 = 2x + 10$ is also 3. You should check this to show that we have made no mistake.

Now we can find the solution of an equation like

$$4x + 5 = 3x + 2.$$

We wish to find an equivalent equation in which x occurs on one side only. To do this we can add $^-(3x)$ to both sides to get

$$^-(3x) + 4x + 5 = {}^-(3x) + 3x + 2.$$

This results in the sentence

$$x + 5 = 2.$$

(You should fill in the steps needed to show this.) Then, if we add $^-5$ to both sides of this equation (since we want to have x by itself on one side) we have

$$\begin{aligned}(x + 5) + {}^-5 &= {}^-5 + 2\\ x + (5 + {}^-5) &= {}^-5 + 2\\ x + 0 &= {}^-3\\ x &= {}^-3\end{aligned}$$

Thus $x = {}^-3$ is equivalent to the equation $4x + 5 = 3x + 2$, which shows that ${}^-3$ is its solution.

The solution of the equation $x = {}^-3$ is obvious and, since it is equivalent to the equation $4x + 5 = 3x + 2$, this equation also has the solution ${}^-3$. In fact, a method of solving an equation is to find an equivalent equation which has an obvious solution--that is, of the form $x =$ some number. Let us go back over the process we used. We first added ${}^-(3x)$ to both sides of the equation in order to get an equivalent equation in which x occurred on only one side: $x + 5 = 2$. Then, since we wanted an equation of the form, $x =$ some number, we added ${}^-5$ to both sides.

To make sure that we have made no mistake, let us check to see that ${}^-3$ is really a solution of the equation:

$$4x + 5 = 3x + 2.$$

If $x = {}^-3$, the left side of the equation becomes

$$({}^-3) \cdot 4 + 5 = {}^-12 + 5 = {}^-7.$$

If $x = {}^-3$, the right side of the equation becomes

$$3({}^-3) + 2 = {}^-9 + 2 = {}^-7.$$

Hence for this value of x, the number on the left side is equal to that on the right. This is our check.

Of course there are equations which have no solutions. One such equation is $x + 3 = x$. This may be considered to be obvious since no number can be 3 greater than itself. But let us find what this equation is equivalent to. We may add ${}^-x$ to both sides and have

$${}^-x + (x + 3) = {}^-x + x$$

or

$$({}^-x + x) + 3 = {}^-x + x$$
$$0 + 3 = 0$$
$$3 = 0.$$

(Remember that just as ${}^-3$ is the number with the property that ${}^-3 + 3 = 0$, so ${}^-x$ is the number with the property that ${}^-x + x = 0$.)

So the given equation is equivalent to $3 = 0$. This has no solution and hence the given equation has no solution.

Other equations have many solutions. Consider $2x = x + x$. This is true for all values of x. You might like to show that this is equivalent to $0 = 0$.

Exercises 2-3a

1. Using the methods of the previous section, find four equations equivalent to each of the following equations:

   (a) $x + 7 = 13$

   (b) $17 = x + {}^{-}3$

   (c) $x = 7$

2. Use the addition property to solve the following equations. Check your results.

   Example: $x + ({}^{-}3) = 11$. First use the addition property and add 3 to both sides of the equation. This gives

   $$(x + {}^{-}3) + 3 = 11 + 3.$$

   By the associative property of addition this is equivalent to

   $$\begin{aligned} x + ({}^{-}3 + 3) &= 14, \\ x + 0 &= 14 \\ x &= 14. \end{aligned}$$

   To check this see that if $x = 14$, $x + {}^{-}3$ is $14 + {}^{-}3$ which is equal to 11.

   (a) $x + 5 = 6$

   (b) $x + 6 = 5$

   (c) $x + {}^{-}7 = 7$

   (d) $x + {}^{-}7 = {}^{-}7$

   (e) $t + 6 = {}^{-}13$

   (f) $4 = x + 3$

   (g) ${}^{-}2 = {}^{-}4 + x$

   (h) $x + \frac{3}{2} = 10$

   (i) $y + {}^{-}(\frac{3}{2}) = \frac{5}{2}$

   (j) $u + 14 = \frac{9}{5}$

   (k) $\frac{13}{7} = 1 + x$

   (l) $x + {}^{-}(\frac{4}{9}) = {}^{-}(\frac{7}{13})$

3. Apply the addition property to these equations, adding the indicated number, and write the resulting equation.

Example: $3x + 4 = 5$ (add $^-4$)
$(3x + 4) + {}^-4 = 5 + {}^-4$ by the addition property.
$3x + (4 + {}^-4) = 1$ by the associative property.

The resulting equation is: $3x = 1$.

(a) $2x + 5 = 10$ (add $^-5$)
(b) $3x + 10 = 5$ (add $^-10$)
(c) $5x + 2 = {}^-2$ (add $^-2$)
(d) $10x + {}^-1 = 9$ (add 1)
(e) $2u + 1 = 11$ (add $^-1$)
(f) $2x + {}^-3 = 9$ (add 3)
(g) $^-4y + {}^-3 = \frac{9}{2}$ (add 3)

4. (a) What number do you add (using the addition property) to solve $x + 3 = 2$?

(b) What number do you add (using the addition property) to solve $x + ({}^-7) = 4$?

(c) What is the relation between 3 and $^-3$ relative to addition?

(d) What is the relation between 7 and $^-7$ relative to addition?

5. (a) If $x = 3$, what is ^-x?

(b) If $x = \frac{1}{2}$, what is ^-x?

(c) If $x = {}^-3$, what is ^-x? Answer: By ^-x, we mean that number with the property that $x + {}^-x = 0$. So for $x = {}^-3$, we have

$$^-3 + ({}^-x) = 0.$$

But we know that $^-3 + 3 = 0$. Hence ^-x must be equal to 3. In other words, $^-({}^-3) = 3$.

(d) If $x = {}^{-}(\frac{1}{2})$, what is ${}^{-}x$?

(e) If $x = {}^{-}7$, what is ${}^{-}x$?

6. Earlier in this section we solved the equation

$$4x + 5 = 3x + 2$$

by adding ${}^{-}(3x)$ to both sides of the equation. Suppose we had begun by adding ${}^{-}(4x)$ to both sides. Could the solution be found this way? Do you think this is a simpler method of solution?

7. Simplify each of the following:

(a) ${}^{-}x + 3x = ?$

(b) $3x + {}^{-}x = ?$

(c) ${}^{-}(2x) + 3x = ?$

8. Solve the equation: $2x + 7 = x$.

9. Solve the equation: $2x + 3 = x + 2$.

10. Solve the equation: $x = 2x + 6$.

11. Solve the equation: $3x + 5 = 2x$.

Adding the same number to both sides of an equation is not the only way to get an equivalent one. We may also multiply both sides by the same number. For example:

$$5 + 2 = 7,$$

is a true sentence. If we multiply both sides by 3, we get

$$3(5 + 2) = 3 \cdot 7.$$

This is also true. This is an example of the *multiplication property of equality*.

The Multiplication Property of Equality: *If* a *and* b *are two equal numbers, then* $ca = cb$.

Is it true that $ca = cb$ implies that $a = b$? Before answering this question, let us consider the solution of an equation using the multiplication property. Since our method of solution is very much like that for an equation involving addition, in the left-hand column below we shall solve an equation using the addition property and in the right-hand column a similar equation using the multiplication property.

Problem: Solve $3 + x = 6$

First add the additive inverse of 3, to both sides of the equation to get

$$^-3 + (3 + x) = {}^-3 + 6.$$

Use the associative property of addition:

$$\begin{aligned}(^-3 + 3) + x &= {}^-3 + 6\\ 0 + x &= 3\\ x &= 3\end{aligned}$$

Problem: Solve $3x = 6$

First multiply both sides of the equation by $\frac{1}{3}$, the multiplicative inverse of 3, to get

$$\frac{1}{3}(3x) = \frac{1}{3}(6).$$

Use the associative property of multiplication:

$$\begin{aligned}(\frac{1}{3} \cdot 3)x &= \frac{1}{3} \cdot 6.\\ 1 \cdot x &= 2\\ x &= 2\end{aligned}$$

We can use this same parallel treatment to show that

$$\text{if } ca = cb, \text{ and } c \neq 0, \text{ then } a = b.$$

Problem: Prove that,

if $c + a = c + b$, then $a = b$.

First add ^-c, the additive inverse of c, to both sides of the equality to get

$$^-c + (c + a) = {}^-c + (c + b)$$

Using the associative property of addition, we have

$$\begin{aligned}(^-c + c) + a &= (^-c + c) + b\\ 0 + a &= 0 + b\\ a &= b\end{aligned}$$

Problem: For $c \neq 0$ prove that,

if $ca = cb$, then $a = b$.

First multiply both sides by $\frac{1}{c}$, the multiplicative inverse of c, (note that this inverse exists only if $c \neq 0$) to get

$$\frac{1}{c} \cdot (ca) = \frac{1}{c} \cdot (cb)$$

Using the associative property of multiplication, we have

$$\begin{aligned}(\frac{1}{c} \cdot c)a &= (\frac{1}{c} \cdot c)b\\ 1 \cdot a &= 1 \cdot b\\ a &= b\end{aligned}$$

Thus we have shown that if $c \neq 0$, $ca = cb$ and $a = b$ are equivalent equations. This means that if $c \neq 0$ and $ca = cb$, then $a = b$; also if $c \neq 0$ and $a = b$, then $ca = cb$.

Now let us apply this result to the solution of the equation,

$$3x + 1 = 13.$$

We wish to have $3x$ by itself on one side of the equation and, as in the first part of this section, we can accomplish this by adding $^{-}1$ to both sides. This gives us

$$\begin{aligned} 3x + 1 + {}^{-}1 &= 13 + {}^{-}1, \\ 3x + 0 &= 12, \\ 3x &= 12. \end{aligned}$$

Since we wish to have an equation of the form $x =$ some number, we can accomplish this by multiplying both sides by $\frac{1}{3}$, the reciprocal (that is, the multiplicative inverse) of 3. Thus the equation

$$3x = 12$$

is equivalent to $$\frac{1}{3} \cdot (3x) = \frac{1}{3} \cdot (12),$$

or $$(\frac{1}{3} \cdot 3) \cdot x = 4,$$

or $$x = 4.$$

We have shown that the equation $3x + 1 = 13$ is equivalent to $x = 4$. Since $x = 4$ has the obvious solution 4, so is 4 the solution of $3x + 1 = 13$. From what we have done, we can be sure that if we did not make a mistake $x = 4$ is the solution of the equation $3x + 1 = 13$. But it is reassuring and is also good policy for us to check this answer to see if it is indeed a solution. If we replace x by 4 in $3x + 1$ we get $3 \cdot 4 + 1$ which is equal to 13. This is the check we wanted.

Class Exercises 2-3c

1. Indicate which property, the addition or the multiplication property of equality, and which number is to be added or used as a multiplier in solving the following equations.

(a) $x + 10 = 22$

(b) $6.2 + x = 1.12$

(c) ${}^{-}2 + x = {}^{-}10$

(d) $5x = 15$

(e) $6 = \frac{x}{18}$

(f) $14 - x = 0$

(g) $\frac{1}{2}x = 17$

(h) $18 + y = 8.6$

(i) $y + 6 = 5 + 3$

(j) $0.08d = 73$

(k) $19 = 6 - y$

(l) $\frac{2}{3}n = 15 + 0.4$

(m) $45 \cdot b = 1$

(n) $\frac{7}{c} = 1,\ c \neq 0$

2. Find the reciprocals of each of the following numbers:

(a) 7. Answer: $\frac{1}{7}$ is the reciprocal since $7(\frac{1}{7}) = 1$.

(b) 5

(c) ${}^{-}3$

(d) $\frac{1}{2}$. Answer: the reciprocal of a number b is defined to be that number $\frac{1}{b}$ such that $b(\frac{1}{b}) = 1$. Since $\frac{1}{2}(2) = 1$, 2 is the reciprocal of $\frac{1}{2}$.

(e) $\frac{1}{4}$

(f) $\frac{2}{3}$

(g) ${}^{-}(\frac{1}{2})$

3. Find the additive inverse and the multiplicative inverse of each of the following, being careful to state which is which:

(a) 3 (b) $\frac{1}{2}$ (c) $\frac{5}{7}$

Exercises 2-3b

1. Find the additive inverse and the multiplicative inverse of each of the following, being careful to state which is which:

(a) 7

(b) $\frac{3}{4}$

(c) ${}^{-}2$

(d) $\frac{1}{2}$

2. Solve each of the following equations and state where you use the addition property and where the multiplication property of equality.

   (a) $3x + 2 = 14$

   (b) $7x = 2$

   (c) $^{-}3x + 7 = 22$

   (d) $\frac{1}{2}x = 7$

   (e) $^{-}(\frac{1}{2})x = 14$.

Let us try our methods on a more complicated equation:

$$^{-}(\tfrac{1}{2})x + 2 = 2x + \tfrac{1}{2}.$$

We wish first to find an equivalent equation in which only one side has a term in x. Here we use the addition property and add $\frac{1}{2}x$ to both sides:

$$\tfrac{1}{2}x + \left(^{-}(\tfrac{1}{2})x + 2\right) = \tfrac{1}{2}x + (2x + \tfrac{1}{2}).$$

Using the associative property for addition we have:

$$\text{(A)} \qquad \left(\tfrac{1}{2}x + {}^{-}(\tfrac{1}{2})x\right) + 2 = (\tfrac{1}{2}x + 2x) + \tfrac{1}{2}.$$

Now $\frac{1}{2}x + {}^{-}(\frac{1}{2})x = 0$ since $^{-}(\frac{1}{2})x$ is the additive inverse of $\frac{1}{2}x$. Also by the distributive property,

$$\tfrac{1}{2}x + 2x = (\tfrac{1}{2} + 2)x.$$

Hence the equation (A) above is equivalent to

$$0 + 2 = (\tfrac{1}{2} + 2)x + \tfrac{1}{2};$$

that is,

$$2 = \tfrac{5}{2}x + \tfrac{1}{2}.$$

Since we wish the term in x to be by itself on one side of the equation, we can again use the addition property and add $^{-}(\frac{1}{2})$

to both sides of the equation to get

$$\frac{3}{2} = \frac{5}{2}x.$$

Since we want an equation of the form, a number = x, we can use the multiplication property and multiply both sides by $\frac{2}{5}$, the reciprocal of $\frac{5}{2}$, and get

$$\frac{2}{5} \cdot \frac{3}{2} = \frac{2}{5} \cdot \frac{5}{2}x$$

We have, finally (you will need to write in some steps),

$$\frac{3}{5} = x.$$

Thus $\frac{3}{5}$ is the solution of the equation ${}^-(\frac{1}{2})x + 2 = 2x + \frac{1}{2}$.

Now getting this result was rather long and there were many opportunities for mistakes. We should check our result.

If $x = \frac{3}{5}$, the left side of the given equation, ${}^-(\frac{1}{2})x + 2$, becomes ${}^-(\frac{1}{2}) \cdot \frac{3}{5} + 2 = {}^-(\frac{3}{10}) + 2 = {}^-(\frac{3}{10}) + \frac{20}{10} = \frac{17}{10}$

If $x = \frac{3}{5}$, the right side of the given equation, $2x + \frac{1}{2}$ becomes $2 \cdot \frac{3}{5} + \frac{1}{2} = \frac{6}{5} + \frac{1}{2} = \frac{12}{10} + \frac{5}{10} = \frac{17}{10}$

This shows that for $x = \frac{3}{5}$, ${}^-(\frac{1}{2})x + 2$ is equal to $2x + \frac{1}{2}$.

There are other methods of solving this equation. One would be to begin by multiplying both sides of the equation by 2, to get rid of the fractions. You are asked to try this out in an exercise below.

Class Exercises 2-3d

1. What property is used, and how is it used, to get the second equation from the first?

Example: (1) $2x + 4 = 7$
(2) $2x = 3$ addition property, adding ${}^-4$.

(a) (1) $(\frac{1}{8}k) + 1 = 1$
(2) $\frac{1}{8}k = 0$

(b) (1) $1.6 = 4y$
(2) $0.4 = y$

(c) (1) $\frac{2(m + 5)}{3} = 6$
(2) $2(m + 5) = 18$

(d) (1) $^{-}x = 5$
(2) $x = {}^{-}5$

(e) (1) $2(y + 4) = 8$
(2) $y + 4 = 4$

(f) (1) $\frac{2}{5}x = 10$
(2) $\frac{1}{5}x = 5$

(g) (1) $(0.3m) - 7.2 = 5$
(2) $(3m) - 72 = 50$

*(h) (1) $\frac{4}{n} = {}^{-}26$
(2) $4 = {}^{-}26n$

*(i) (1) $5x - 2 = 3x + 6$
(2) $2x - 2 = 6$

2. Solve and check each of the following equations:

(a) $3y + {}^{-}2 = 7$
(b) $7 = 3x + 1$
(c) $6 = 3w$
(d) $\frac{1}{2}t - 1.7 = {}^{-}1.3$
(e) $2 = \frac{x}{18}$
(f) $0.14 + x = 5.28$
(g) $5x - 7 = 2x$
(h) $x = 7 - 2x$

Exercises 2-3c

1. Solve the following equations by using the properties of "equals." Give your reason for each step.

(a) $2x + 1 = 7$
(b) $y - 2 = 6$
(c) $\frac{t}{2} - 3 = {}^{-}4$
(d) $3x - 5 = {}^{-}4$

2. Solve the following equations.

(a) $x + 3 = 5$
(b) $3 + y = {}^{-}5$
(c) $2v + 3 = 5$
(d) $3 + 2m = {}^{-}5$
(e) $y - 3 = 5$
(f) $3 - u = {}^{-}5$
(g) $2w - 3 = 5$
(h) $3 - 2s = {}^{-}5$

*(i) $2w + 7 = 5w + 1$ *(k) $2t - 11 = 5t + 1$

*(j) $15 + 2w = {}^{-}5w + 1$ *(l) $15 - 5w = 2w + 1$

3. (a) In solving the equation $9x = 27$ what number would you use as a multiplier?

   (b) In solving the equation $\frac{1}{3}x = 4$ what number would you use as a multiplier?

   (c) In solving the equation $\frac{4}{5}x = \frac{1}{2}$ what number would you use as a multiplier?

   (d) What is the relation between 9 and $\frac{1}{9}$, relative to multiplication?

   (e) What is the relation between 3 and $\frac{1}{3}$, relative to multiplication?

   (f) What is the relation between $\frac{4}{5}$ and $\frac{5}{4}$, relative to multiplication?

4. In solving an equation such as $3x + 1 = 9$, you have learned to use the addition property first (to find $3x$) and the multiplication property second (to find x). Sometimes you will find it best to reverse the order in which you use these properties. Solve the following equations by using the multiplication property first.

   (a) $4(x + 1) = 12$ (d) $0.6(x - 0.3) = 0.2$

   (b) $7(x - 2) = 13$ (e) $\frac{3x + 4}{2} = 7$

   (c) $\frac{x + 9}{3} = 5$ (f) $\frac{4x + 1}{0.12} = 3$

5. To solve the equation $\frac{1}{2}({}^{-}x) + 2 = \frac{1}{2} + 2x$, begin by multiplying both sides by 2 and get the equivalent equation

$${}^{-}x + 4 = 1 + 4x.$$

Then find the solution of this equation. Does it agree with that in the previous discussion? Do you think this method is easier than the one used?

2-4. Solving Inequalities

In Section 2, we found by inspection solutions of various inequalities. The methods which we used to solve equalities in the previous section may be used for inequalities as well. To show the similarity, let us solve an equation and a related inequality.

To solve: $x + 4 = 7$

Add ${}^{-}4$ to both sides of the equation, using the addition property of equality:

$$(x + 4) + {}^{-}4 = 7 + {}^{-}4$$

Using the associative property of addition,

$$x + (4 + {}^{-}4) = 7 + {}^{-}4$$
$$x + 0 = 3$$
$$x = 3.$$

To solve: $x + 4 < 7$

Add ${}^{-}4$ to both sides of the inequality, using the "addition property of inequality"

$$(x + 4) + {}^{-}4 < 7 + {}^{-}4$$

Using the associative property of addition,

$$x + (4 + {}^{-}4) < 7 + {}^{-}4$$
$$x + 0 < 3$$
$$x < 3.$$

We know that each equation on the left is equivalent to all the others. We assumed on the right that the same statements could be made about the inequalities. We need to show the addition property of inequality:

The Addition Property of Inequality: If $a < b$, then $a + c < b + c$.

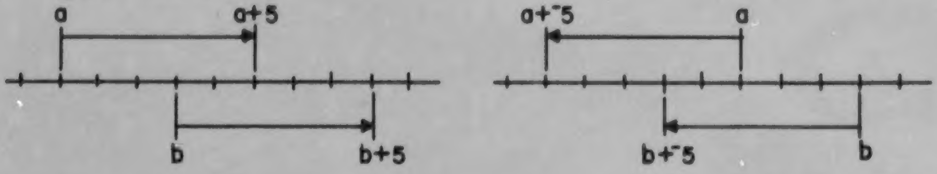
To show this, first suppose $c = 5$. Then $a < b$ means that on the horizontal number line, the point which a represents is to the left of the point which b represents. Now the point which $a + 5$ represents is 5 units to the right of the point which a represents; the point which $b + 5$ represents is 5 units to the right of the point which b represents. Hence the point which $a + 5$ represents is to the left of the point which $b + 5$ represents. This means

$$a + 5 < b + 5.$$

Second, if c were ${}^{-}5$, the point which $a + {}^{-}5$ represents is 5 units to the left of the point which a represents, and similarly for $b + {}^{-}5$. Again

$$a + {}^{-}5 < b + {}^{-}5.$$

See the figure below:



In general, if c is a positive number, the point represented by $a + c$ is c units to the right of the point represented by a and similarly for $b + c$ and b. This, if c is a positive number,

$$\text{and } a < b \text{ then } a + c < b + c$$

Why would the same result hold if c were a negative number? Thus we have shown the addition property of inequality.

Just as in the case of equality we can show:

> If $a < b$, then $a + c < b + c$ and if $a + c < b + c$, then $a < b$.

This shows that if we add the same number to both sides of an inequality, we have an equivalent inequality. For instance the inequality

$$x + {}^{-}3 < 8$$

is equivalent to the inequality $(x + {}^{-}3) + 3 < 8 + 3$, that is,

$$x < 11.$$

Exercises 2-4

1. Find the set of solutions of each of the following inequalities:

(a) $x + 5 < 7$	(d) $y + {}^{-}3 < 10$
(b) $7 > x + 5$	(e) $10 < y + {}^{-}3$
(c) $x + {}^{-}2 < 8$	(f) $2x + 3 < x + 2$

(g) $x + 4 < 5 + x$

(h) $x + 4 > 5 + x$

(i) $3x + 2 < 2x + {}^{-}3$

(j) $\frac{1}{2}x + 3 > {}^{-}(\frac{1}{2})x + 4$

(k) $7 + 2x < x + {}^{-}(\frac{1}{7})$

2. Show that if c is a negative number and if $a < b$, then $a + c < b + c$.

3. Use the addition property of inequality to show that if $a + c < b + c$, then $a < b$. (This may be done by the same method which we used for equalities.)

4. If $a > b$, is it true that $a + c$ must be greater than $b + c$? If so, show why. If not, show why not.

5. If $a \neq b$ must it be true that $a + c \neq b + c$? Give reasons for your answer.

6. If $a < b$ must it be true that $2a < 2b$? Show why this is true.

*7. If $a < b$, must it be true that $({}^{-}2)a < ({}^{-}2)b$? Why or why not?

2-5. Number Sentences with Two Unknowns

In the previous examples of number phrases and number sentences, there was only one unknown number. We could also have more than one unknown number. Look at this sentence:

$$x + 1 = y.$$

If $x = 3$ and $y = 5$, is the sentence true or false? If $x = 7$, what must y be for the sentence to be true? If $y = {}^{-}6$, then we have $x + 1 = {}^{-}6$. What must x be in order for this sentence to be true? How did you learn to solve an equation like $x + 1 = {}^{-}6$ in Section 2-3?

Each solution of the equation $x + 1 = y$ is a pair of numbers. We can make a table listing some of these pairs:

Table for

$x + 1 = y$

x	y
0	1
1	
2	
$^-(\frac{2}{3})$	
	$^-(\frac{13}{3})$

Before you continue reading, copy this table and work out the missing numbers. For example, to fill in the third line, replace x by 2 in the above equation. Ask yourself, "What are the possible values of y?"

You should read the rest of this chapter with pencil and paper handy. Do not go on to a new paragraph until you have answered all the questions in the paragraph you have just read. In much of this section you will find it convenient to use graph paper and a ruler, too.

If $x = 0$ and $y = 1$, then the equation $x + 1 = y$ is true. Hence, we say that the pair $(0, 1)$ is a solution of the equation. Notice that it makes a difference which number is named first. The pair $(1, 0)$ is not a solution since if $x = 1$ and $y = 0$, then

$$x + 1 = 1 + 1 = 2 \quad (\text{not } 0)$$

so that the equation $x + 1 = y$

is not true.

You remember from Chapter 1, that a pair in which the objects are considered in a definite order is called an ordered pair.

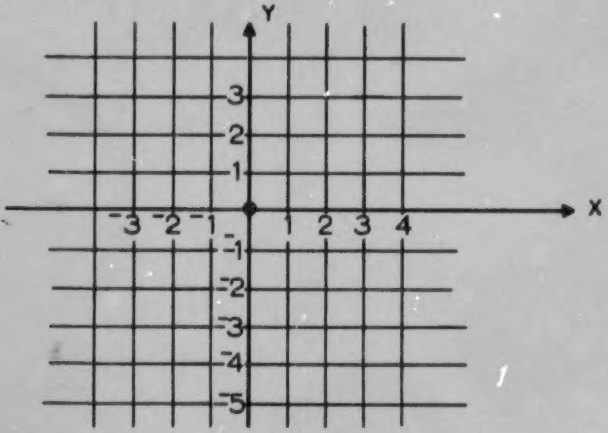
The ordered pair $(2, 7)$ is the same as the ordered pair (x, y) if $x = 2$ and $y = 7$, and only then. This pair is different from the ordered pair $(7, 2)$.

The solution set of the above sentence

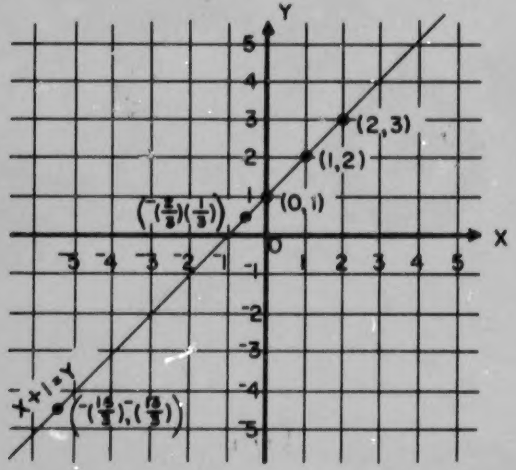
$$x + 1 = y$$

is a set of ordered pairs of numbers. For what number y is the ordered pair $(2, y)$ in the solution set?

In order to picture the solution set on your graph paper, pick out two lines for the X-axis and the Y-axis as in Chapter 1, and draw them in heavily with your pencil. Label the vertical and horizontal lines as shown.



Mark off on your graph paper the points $(0, 1)$, $(1, 2)$, etc., whose coordinates are in the solution set of $x + 1 = y$. What do you notice about them? All of them lie on a simple geometric figure. To what set of points does the solution set correspond? In Chapter 1 you learned to call it the graph of the given number sentence, or equation. The graph of $x + 1 = y$ is shown here:



Let us try another example. What is the solution set of the equation

$$2x + y = {}^{-}1?$$

If we give x a certain value, we obtain an equation to solve for y. If we try a different value for x we obtain a different equation to solve for y. Similarly, if we try different values for y, we obtain different equations to solve for x.

For example: Let $x = {}^{-}3$. This gives the equation

$$2({}^{-}3) + y = {}^{-}1$$
$${}^{-}6 + y = {}^{-}1.$$

We solve this equation by the methods we learned in Section 4.

$6 + ({}^{-}6 + y) = 6 + {}^{-}1$ by the addition property,

$(6 + {}^{-}6) + y = 5$ by the associative property of addition

so $y = 5.$

If $y = 5$,

then ${}^{-}6 + y = {}^{-}6 + 5 = {}^{-}1$

so ${}^{-}6 + y = {}^{-}1.$

Thus, 5 is a solution, the only solution.

Thus, $({}^{-}3, 5)$ is a solution of the equation $2x + y = {}^{-}1$.

Follow this example to complete the table of solutions of $2x + y = {}^{-}1$ below. Perhaps you can do some of the steps in your head.

x	y
$^{-}3$	5
$^{-}1$	
0	
4	
	$^{-}2$
	0
	2

When this table is completed we have seven ordered pairs of numbers which are in the solution set of the equation $2x + y = {}^-1$. Choose an X-axis and a Y-axis on your graph paper and locate the points whose coordinates are these ordered pairs. Do all of them seem to lie on a line? Draw the line. Locate a point on the line which is not one of the seven which you have plotted. Can you find the coordinates of this point by measuring certain distances? The coordinates form an ordered pair of numbers. If your drawing and measurement were perfectly accurate, this ordered pair would also be in the solution set of the equation. Is it?

From the examples which you have seen in this section and in Chapter 1, you have perhaps guessed that the graph of any equation of the form

$$ax + by = c,$$

where a, b, and c are known numbers, lies on a line. This is true. For that reason we usually call an equation of this type a linear equation.

Exercises 2-5a

1. Make up a table showing some of the ordered pairs from the set of solutions for each of the following equations. On the same set of axes draw graphs of each of the equations.

   $y = x + 1, \quad y = 2x + 1, \quad y = 3x + 1, \quad y = {}^-2x + 1.$

2. Do the same for $y = x + 1$, $y = x + 2$, $y = x + {}^-3$.
3. Do the same for $x + y = 0$, $x + y = 1$, and $x + y = {}^-1$.
4. Do the same for $\frac{x}{2} + \frac{y}{3} = 1$ and $\frac{x}{3} - \frac{y}{2} = 1$.
5. Do the same for $y = x + 1$ and $x + y = 1$.
6. Do the same for $y = 2x + 3$ and $y = {}^-(\frac{1}{2}x) + 3$.

Let us try another problem in which two unknowns are involved. What are the possible lengths of the sides of a rectangle if the perimeter is 16 inches? If we let the lengths of two adjacent sides be called x inches and y inches then we must have

$$2x + 2y = 16.$$

We must be careful, however! This equation is not a complete translation of the real situation into mathematical language. Can the length of a side of a rectangle be a negative number of inches? Can it be zero inches? We must have $x > 0$ and $y > 0$, and the number sentence which really describes the situation is this:

$$2x + 2y = 16 \text{ and } x > 0,\ y > 0.$$

We can find several ordered pairs in the solution set. Which of the following pairs are solutions?

$$(^-1, 9),\ (1, 7),\ (2, 6),$$
$$(\tfrac{7}{2}, \tfrac{9}{2}),\ (5, 3),\ (7, 2),$$
$$(8, 0),\ (9.5, {}^-1.5)$$

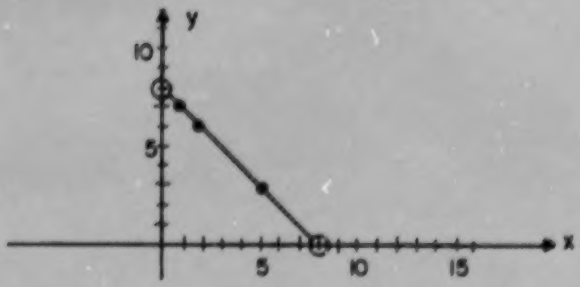
Remember that to be a solution, an ordered pair must make the entire number sentence,

$$2x + 2y = 16 \text{ and } x > 0,\ y > 0,$$

true.

The graph of the equation part of the above number sentence lies on a line. Sketch it. The second part of the sentence, namely, "$x > 0$, $y > 0$," says that the point corresponding to conditions stated above must lie in which quadrant? The graph of the number sentence is the part of the graph of $2x + 2y = 16$ that lies in the first quadrant.

The graph is a segment with its endpoints removed as shown here:



Which of the above solutions of the number sentence have been plotted on the graph in the figure?

In this example we had a number sentence which was built of several shorter number sentences, one of which was an equation. Let us try another number sentence of this type.

Bonnie has in her purse 3 dollars in dimes and quarters. What possible combinations can she have?

Let d be the number of dimes and q be the number of quarters. Just as the value of 3 dimes is $10 \cdot 3$ cents, so the value of d dimes is $10 \cdot d$ cents. The total value of these coins is $(10d + 25q)$ cents, and this must be equal to 3 dollars. But, wait a minute! We must make up our minds whether we want to measure our money in cents or dollars. Let us use cents throughout. Then, 3 dollars is 300 cents. Therefore, the pair (d,q) is a solution of the equation

$$10d + 25q = 300.$$

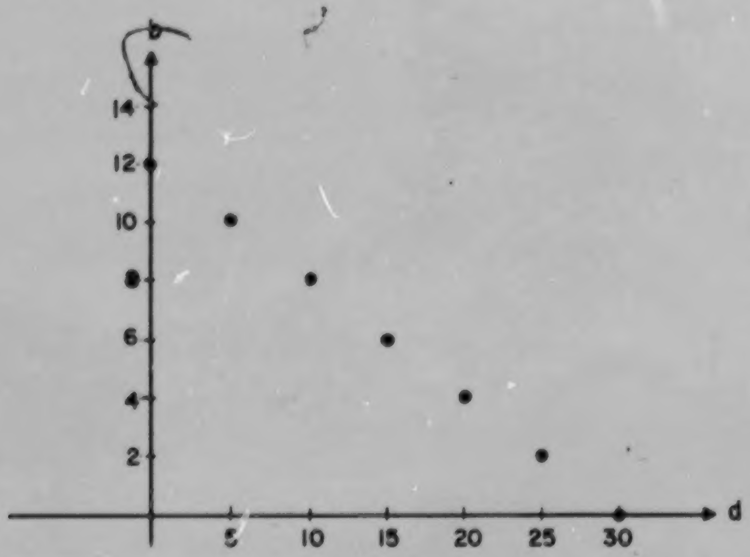
Again we must be careful! Bonnie cannot have twenty-seven and one-half dimes, nor can she have $^-3$ dimes. The unknown numbers in this problem must be non-negative integers. The number sentence which really describes this situation is:

$10d + 25q = 300$, and d and q are non-negative integers.

The solution set of this number sentence is made up of the seven ordered pairs:

$(0, 12)$, $(5, 10)$, $(10, 8)$, $(15, 6)$,
$(20, 4)$, $(25, 2)$, $(30, 0)$.

The graph of this number sentence consists of only seven points.



You may disagree with the above solution on one point, since there is another way to interpret the statement, "Bonnie has in her purse 3 dollars in dimes and quarters." Does this include the possibility of her having no dimes and 25 quarters? Some people would say "yes" and others would say "no." If your answer is "no," two solutions would be excluded: no dimes and 25 quarters, and 30 dimes and no quarters. These are represented by the ordered pairs: $(0, 12)$, and $(30, 0)$.

We have found the graphs of certain equations. Now let us see what the graph of an inequality looks like. To do this, first compare the table of solutions of $y = x + 1$, which we have already found with the table of solutions of $y > x + 1$.

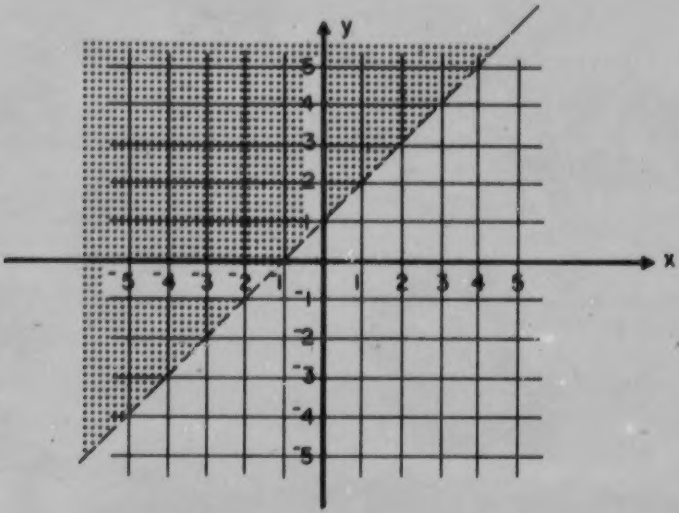
$y = x + 1$

x	y
0	1
1	2
2	3
$^{-}1$	0

$y > x + 1$

x	y
0	> 1
1	> 2
2	> 3
$^{-}1$	> 0

This shows that for $x = 1$, for instance, y can be any number greater than 2; not only can y be 3, 4, 5, and so forth, but also $2\frac{1}{2}$, $2\frac{1}{4}$, 2.1. Thus the graph of $y > x + 1$ is as shown in the figure. The graph of the inequality does not include the line itself. In such cases the line is shown as a broken or "dotted" line.

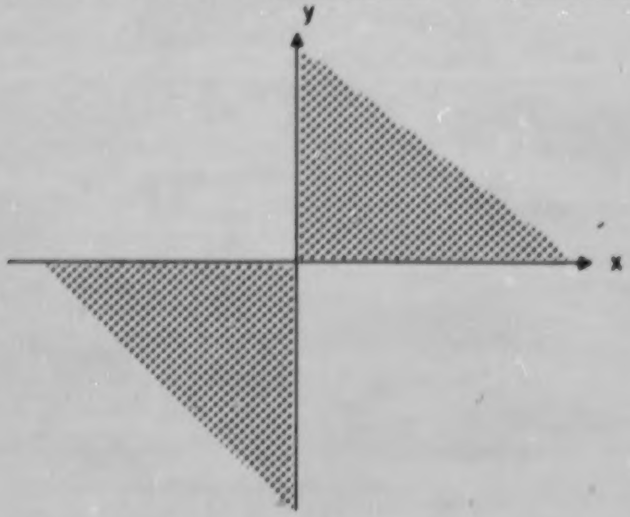


(The dotted line is the graph of $y = x + 1$. No points from this line are in the graph of $y > x + 1$.)

What is the graph of the following inequality with two unknown numbers?

$$xy > 0$$

The inequality says that the product of x and y must be positive. What do you know about two numbers whose product is a positive number? Can either one be zero? Can x be positive and y negative in a pair (x, y) which is a solution? The pair (x, y) is a solution if both x and y are __________ or if both x and y are __________. You fill in the blanks. The graph of this inequality is, then, the entire __________ and __________ quadrants.



Let us consider one equation which is not a linear equation. Consider

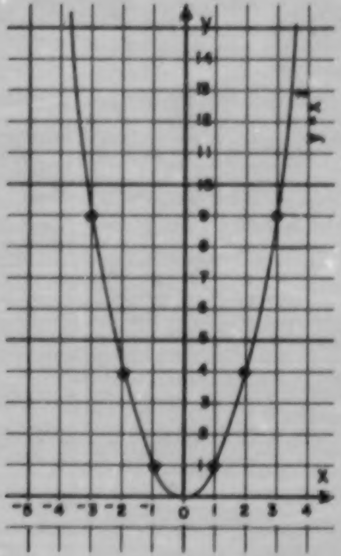
$$y = x^2.$$

If we take a known value of x in this equation, the resulting equation in the unknown y is not hard to solve. Fill in the table of values.

x	y
$^{-}4$	16
$^{-}3$	
$^{-}2$	
$^{-}1$	1
0	

x	y
0	
1	1
2	
3	
4	

Plot these points on your graph paper. Then sketch the graph of the equation.

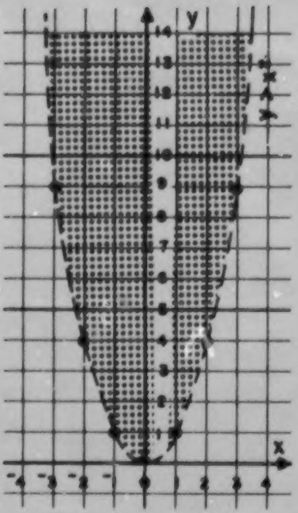


This curve is called a parabola. It is a very important curve which occurs in many ways in problems about natural events. The points on this curve are the points whose coordinates (x, y) are solutions of $y = x^2$. All points with coordinates (x, x^2), where x stands for any number, lie on the parabola. Where will a point with coordinates (x, y) lie if $y > x^2$? If a point (x, y) lies above the parabola what can you say about y and x^2? Which must be greater?

The solution set of the number sentence

$$y > x^2$$

is the set of all ordered pairs (x, v) for which $y > x^2$, and the graph of this open sentence is contained in the region of the plane above the parabola which we sketched above. It does not include the curve itself. The following figure is the graph of $y > x^2$.



Exercises 2-5b

1. (a) Draw the graphs of the following equations on the same set of axes: $y = x^2$ and $y = {}^-(x^2)$.

   (b) Do the same for $y = x^2$ and $y^2 = x$.

   (c) Do the same for $xy = 1$, $xy = {}^-1$, and $xy = 0$.

2. Sketch the graphs of the following:

   (a) $x + y = 1$ and $x > 0$, $y > 0$.

   (b) $x + y = 10$ and x and y are positive integers.

   (c) $y = x^2$ and $x < {}^-1$.

   (d) $y = 1$. (Hint: this is the same as $y = 1 + (0 \cdot x)$.

   (e) $y^2 = 1$.

   (f) $x = 1$.

   (g) $x^2 = 0$.

   (h) $x = 0$ and $y = 0$.

(i) $x^2 + y^2 = 0$.

*(j) y = the larger of the numbers $x + 1$ and $2 - x$.

*(k) $y = x$ when $x \geq 0$ and $y = {}^-x$ when $x < 0$.

3. Consider the number sentence

"________, and x and y are
non-negative integers,"

with each of the following equations filling in the blank. List the solution set in each case, and write the number of solutions which the sentence has.

(a) $x + y = 1$.

(b) $x + y = 2$.

(c) $x + y = 20$.

(d) $x + 2y = 0$.

(e) $x + 2y = 1$.

(f) $x + 2y = 2$.

(g) $x + 2y = 3$.

(h) $x + 2y = 4$.

(i) $x + 2y = 25$.

(j) $5x + 7y = 35$.

(k) $5x + 7y = 36$.

(l) $5x + 7y = 37$.

4. A chain store has 5 tons of coffee in its warehouse in New Orleans. It sends s tons to San Francisco and n tons to New York. Not all the coffee is sent to either place. The total amount shipped is the entire warehouse supply. Write a number sentence in symbols which describes the relation between s and n. On a pair of axes labeled "s" and "n" draw the graph of this number sentence.

5. Draw the graphs of the following inequalities:

(a) $y < x^2$

(b) $y > 4x^2$

(c) $y < {}^-(x^2)$

(d) $y^2 > x$.

6. In an earlier problem you used the relationship $F = \frac{9}{5}C + 32$ between the temperature reading of a Fahrenheit thermometer and the reading of a Centigrade thermometer placed in the same spot. Draw a pair of axes with the vertical one labeled

F and the horizontal one labeled C. Choose a small enough unit distance and use a large enough piece of paper so that each axis has points $^-50$ and 50. Make a careful drawing of the graph of the above equation. Then answer questions (a) and (b) by measuring certain distances on your drawing.

(a) What is the temperature reading on the Fahrenheit thermometer if the reading on the Centigrade thermometer is $^-25$ degrees? $^-15$ degrees? 0 degrees? 4 degrees?

(b) What is the temperature reading on the Centigrade thermometer if the reading on the Fahrenheit thermometer is $^-30$ degrees? $^-15$ degrees? 0 degrees? 50 degrees?

(c) Check your answers by solving the appropriate equations. Remember that it is impossible to make a perfectly accurate drawing or measurement. By how much did your answers in parts (a) and (b) differ from your answers in this part?

*7. The rate for first-class mail in the United States (according to the act of Congress in 1958) is four cents per ounce or fraction thereof. That is, a letter weighing not more than one ounce costs four cents, a letter weighing more than one ounce but not more than two ounces costs eight cents, and so forth. On a pair of axes with the vertical one labeled c, for the cost in cents, and with the horizontal one labeled w, for the weight in ounces, draw the graph of the cost of first-class mail of all weights up to 6 ounces. The first part of the table is as follows:

w	c
* $w > 0$ and $w \leq 1$	4
$1 < w \leq 2$	8
$2 < w \leq 3$	12

*w may be 1 or any number less than 1, but it must be greater than zero. Similarly, w is greater than 1 and any number less than or equal to two.

8. BRAINBUSTER. Carolyn asked Edward what the temperature in the freezer was. Edward told her, and she asked, "Fahrenheit or Centigrade?" He answered, "Both! The readings are the same." What was the temperature in the freezer?

9. BRAINBUSTER. The income tax law of a certain country can be summarized as follows:

   A person's tax is either (a) $400 less than 20% of his income or (b) zero dollars, whichever is greater.

   (a) Let T be the tax in thousands of dollars on an income of I thousand dollars. Write the number sentence which expresses the relation between T and I.

   (b) Draw two axes with the vertical one labeled T and the horizontal one labeled I. Use $1000 as your unit so that, for example, a distance of 4.850 represents $4850. Draw the graph of the number sentence in (a)

   (c) By measuring the distance on your graph, answer the following questions:

   What is the tax on an income of $10,000?

   What is the tax on an income of $3,500?

   What is the tax on an income of $1,500?

   If a man pays a tax of $1,500 what is his income?

   (d) Check your answers in part (c) by using the number sentence in part (a).

Review Exercises

Write open sentences stating the condition for each of the following problems. Then find the set of solutions for each.

1. A board 45 inches long is to be cut into two pieces so that one piece is 3 inches longer than the other. Find the length of each piece.
2. The width of a rectangle is 10 units less than the length. If the perimeter of the rectangle is 68 units, what are the dimensions of the rectangle?
3. A man left a total of $10,500 for his wife, son, and daughter. The wife's share is $6,000. The daughter received twice as much as the son. How much does the son receive?
4. On Thursday a boy read an additional 15 pages of a novel. At this point he found that he had read less than fifty-one pages. How many pages could he have read prior to Thursday of that week?
5. The manager of a hobby store bought 500 kits of model planes to sell in his store. After selling some of the kits, he took an inventory and learned he had fewer than 100 kits remaining. How many of the 500 kits could have been sold?
6. A student checked the enrollment at her school and learned that during the previous year the number of girls was never the same as the number of boys. How many girls could have been enrolled in the school if the number of boys enrolled was always 176?
7. One number is three times as large as another number. Their difference is 12. What is the smaller number?
8. Dick has $2.73 in pennies, nickels, dimes, quarters and half-dollars. He has the same number of each of the different kinds of coins. How many of each kind of coins does he have?
9. Alice and Don ran for class president. Alice got 30 votes more than Don, but not all 316 students voted. How many votes did Don get?

10. A set of twins find that five years ago their combined ages totaled 18. How old are they now?

11. The number of \$1 bills is five times that of the \$5 bills, and the number of \$10 bills is twice that of the \$1 bills. How many \$5 bills are there if the total number of bills is 48?

12. Let x represent the first of two numbers and let y represent the second number. Write an equation expressing the following conditions.

    (a) The sum of the numbers is 7.

    (b) If the second number is subtracted from the first number the result is 5.

13. (a) Make a table showing some of the ordered pairs from the set of solutions for $x + y = 7$.

    (b) Make a table showing some of the ordered pairs from the set of solutions for $x - y = 5$.

    (c) Draw a graph of each equation in parts (a) and (b), using the same set of axes.

    (d) Note where the lines intersect. What ordered pair is associated with this point?

14. (a) Can you find a member of the set of solutions for the compound sentence "$x + y = 7$ and $x - y = 5$?"

    (b) Is there more than one ordered pair in the set of solutions for the compound sentence in part (a)? Explain.

*15. Find the set of solutions for the compound sentence "$x + y = 0$ and $x - y = 0$."

*16. Find the set of solutions for the compound sentence "$x + 1 = y$ and $x - 1 = y$."

*17. Find three consecutive integers such that the sum of the first and the third is 192.

*18. The sum of the number of degrees in the measures of two congruent angles of a triangle is equal to the number of degrees in the measure of the third angle. What are the measures of the angles?

*19. The Jones family and their neighbors, the Smith family, are going on vacations. The two families will travel in opposite directions. If the Jones family averages 55 miles per hour and the Smith family 45 miles per hour, when will they be 750 miles apart if they start at the same time?

*20. The radar operator on an aircraft carrier detects a contact moving directly toward the carrier. He estimates the distance to the contact at 400 miles and the speed of the contact at 350 miles per hour. How long will it take one of the carrier's planes to intercept the contact if it flies directly toward the contact at 450 miles per hour?

*21. In Problem 20, assume the contact is observed at a distance of 10 miles moving directly away from the carrier. A plane from the carrier starts pursuing the contact. Using the speeds given in Problem 20, how long will it take the carrier plane to overtake the contact? How far will the carrier plane be from its base at the time of contact?

Chapter 3

SCIENTIFIC NOTATION, DECIMALS, AND THE METRIC SYSTEM

3-1. Large Numbers and Scientific Notation

In this text we use small numbers, whenever possible, to make the problems easy for you to do, to make the ideas easy to understand, and to make the homework easy for your teacher to grade! But numbers which arise in everyday situations are often very large or very small. Today's newspaper (June 28, 1961), for example, mentions the 12,500,000 members of a labor federation, $2,484,000,000 in state grants under the National Defense Education Act, and a national debt of $298,000,000,000. You probably enjoy at least 3,888,000 seconds in school each year. No doubt you can think of many other common uses of large numbers. How many times do you think a heart beats in an average lifetime? How far will a satellite travel in 10 years, if it moves at a speed of 50,000 miles per hour?

Numbers as large as a million or a billion now occur frequently in such circumstances as those mentioned above. Actually, we have names for numbers larger than one billion, such as trillion and quadrillion. Consider the numeral 3141592653589793; such a numeral is hard to read when written in this form. One way to make it easier to read is to place a comma to the left of every third digit counting from the right as follows:

3,141,592,653,589,793.

This separates the number of thousands, millions, billions and other major units in a natural way. Although we insert commas from right to left, we read the number from left to right according to the diagram which appears on the following page.

quadrillion			trillion				billion				million				thousand					
one		hundred	ten	one		hundred	ten	one		hundred	ten	one		hundred	ten	one		hundred	ten	one
3	,	1	4	1	,	5	9	2	,	6	5	3	,	5	8	9	,	7	9	3

Thus, we read this number as follows:

Three quadrillion,
one hundred forty-one trillion,
five hundred ninety-two billion,
six hundred fifty-three million,
five hundred eighty-nine thousand,
seven hundred ninety-three.

In reading such a number we have to be careful not to use the word "and." We can see the reason for this if we consider the number 593,000 and how it might be read. If it were read "five hundred and ninety-three thousand," as it is by many people, there might be some misunderstanding. If "and" is associated with addition, the meaning would be 500 plus 93,000. If "and" is interpreted as it is in ordinary English, the meaning would be the two separate numbers, 500 and 93,000. Therefore, it is preferable to read 593,000 as "five hundred ninety-three thousand." Omitting the "and" avoids misunderstanding. We usually do use the "and" to mark the decimal point; e.g., 563.12 is read "five hundred sixty-three and twelve hundredths." This use of "and" does not cause confusion since 563.12 means 563 + 0.12.

Actually, such numbers as these seldom occur. This does not mean that numbers of this size are not used. The point is that we rarely can count precisely enough to use such a number. We would just say that the number counted is about three quadrillion. For example, the population of a city of over a million inhabitants might have been given as 1,[illegible]961, but this just happened to be the sum of the various numbers compiled by the census takers. It is certain that the number changed while the census was being taken, and it is probable that 1,577,000 would be correct to the nearest thousand. For this reason there is no harm in rounding the original number to 1,577,000. In fact, for most purposes, we would merely say that the population of the city is "about one and one-half million," which could be written also:

$$\text{City Population} \approx 1,500,000.$$

The symbol $\approx$ is used to mean "is approximately equal to."

There are other ways of writing such large numbers. Many times there are definite advantages in doing this. A suggestion of one possible way to write a large number is given by our statement "one and one-half million." One million can be written 1,000,000 (but there are lots of zeros in this form!) or $10 \times 10 \times 10 \times 10 \times 10 \times 10$ (even worse, isn't it?) or 10^6 (there, isn't that neat?). The indicated product $10 \times 10 \times 10 \times 10 \times 10 \times 10$ is sometimes read "the product of six tens." When we write it in exponent form, 10^6, the number of 10's used as factors in the product is indicated by the exponent 6 We can also get the exponent 6 by counting the number of zeros in the numeral 1,000,000.

In the same way we write one billion as 1,000,000,000 or 10^9. Thus, the national debt of 298 billion dollars could be written as 298×10^9 dollars. This way of representing large numbers in terms of powers of 10 is developed further in the following class exercises.

Class Exercises 3-1a

1. Write the following as decimal numerals and also in exponent form. Example: one million = 1,000,000 = 10^6.
   (a) one billion (b) one trillion (c) one quadrillion

2. Since $2000 = 2 \times 1000$, we can write 2000 in exponent form as 2×10^3. Using an exponent, write the following:

   (a) 7000
   (b) 50,000
   (c) 3,000,000
   (d) 12,500,000
   (e) 2,484,000,000
   (f) 506,000,000

3. A number like 1500 can be expressed in several ways: 150×10 or 15×10^2 or 1.5×10^3. Similarly 325 can be written as 32.5×10 or 3.25×10^2. Also 298 billion is the same as 298×10^9 or 29.8×10^{10} or 2.98×10^{11}. In each of these examples the last expression is of the form

   (a number between 1 and 10) × (a power of 10).

   Write each of the following in this form.

   (a) 76
   (b) 859
   (c) 7623
   (d) 8,463,000.
   (e) 76.48
   (f) 4832.59
   (g) 841.2
   (h) 9783.6
   (i) 3,412,789.435
   (j) fifty-three billion, six hundred forty-two million, five hundred thousand.

Scientific Notation

As we remarked, we can write 298 billion as 298×10^9 or as 2.98×10^{11}. These are compact ways of writing the number. Also, it is easy to compare several large numbers written in this form. For example, we can tell at a glance that 4.9×10^{13} is bigger than 9.6×10^{12} without counting decimal places in 49000000000000 and 9600000000000. We shall see later on that it often simplifies calculations with large numbers to work with them in such a standard form. This is especially true of computations by slide rule or by logarithms, as you will learn in high school. For these reasons it is common practice in scientific and engineering work to represent numbers in this way, namely in the form

(a number between 1 and 10) × (a power of 10).

The number is then said to be written in scientific notation. If a number is a power of 10 then the first factor is 1 and we usually do not write it. Thus 10000 = $1 \times 10^4 = 10^4$ and 10,000,000 = 10^7 in scientific notation.

Definition. A number is said to be expressed in scientific notation if it is written as the product of a decimal numeral between 1 and 10 and the proper power of 10. If the number is a power of 10, the first factor is 1 and need not be written.

Remark: Sometimes "scientific notation" is referred to as "powers-of-ten" notation. We shall occasionally use this term.

Class Exercises 3-1b

1. (a) Is 15×10^5 in scientific notation? Why, or why not?
   (b) Is 3.4×10^7 in scientific notation? Why, or why not?
   (c) Is 0.12×10^5 in scientific notation? Why, or why not?

2. Write the following in scientific notation:
   (a) 5687 (b) 14 (c) $3\frac{1}{2}$ million

3. Write the following in decimal notation:
   (a) 3.7×10^6 (b) 4.7×10^5 (c) 5.721×10^6

4. Since the earth does not travel in a circular path, the distance from the earth to the sun varies with the time of the year. The average distance has been calculated to be about 93,000,000 miles.
   (a) Write the above number in scientific notation. The smallest distance from earth to the sun would be about $1\frac{1}{2}$% less than the average; the largest distance would be about $1\frac{1}{2}$% more than the average.
   (b) Find $1\frac{1}{2}$% of 93,000,000.
   (c) Find approximately the smallest distance from earth to sun.
   (d) Find approximately the largest distance from earth to sun.

(e) Write the numbers found in Parts (c) and (d) in powers-of-ten-notation.

Note that $146{,}000 = 1.46 \times 10^5$ could also be written as 1.460×10^5 or 1.4600×10^5. Each of these represents 146,000 in scientific notation. Although there are situations in which we wish to write one or more zeros after the "6" in 1.46, we shall not do it in this chapter. However, as you will learn later, in the chapter on relative error, scientists use the first factor in this powers-of-ten notation to indicate the precision with which a quantity has been measured.

Exercises 3-1

1. Write the following in powers-of-ten notation:
   - (a) $1{,}000$
   - (b) $10^1 \times 10^1 \times 10^1 \times 10^7$
   - (c) $10 \times 10 \times 10 \times 10$
   - (d) $10^2 \times 10^7$
   - (e) 10×10^5
   - (f) $10{,}000{,}000$

2. Write the following in scientific notation:
   - (a) $6{,}000$
   - (b) 678
   - (c) $9{,}000{,}000{,}000$
   - (d) $459{,}000{,}000$
   - (e) $78{,}000$
   - (f) 600×10
   - (g) $15{,}600$
   - (h) 781×10^7

3. The total number of stars which can be photographed using present telescopes and cameras is estimated to be about 506,000,000. Write this number in scientific notation.

4. Write a numeral for each of the following in ordinary decimal form (which does not use an exponent or indicate a product):
   - (a) 10^5
   - (b) 5.83×10^2
   - (c) 3×10^4
   - (d) 5.00×10^7
   - (e) 6.3×10^2
   - (f) 8.2001×10^8
   - (g) 436×10^6
   - (h) 17.324×10^5

5. Write the following, using words:
   (a) 783
   (b) 7,500,000
   (c) 632,007
   (d) 362.362
   (e) 4,000,284,632
   (f) 4.2506

6. Round each of the following to the nearest hundred. Express the rounded number in scientific notation.
   (a) 645
   (b) 93
   (c) 1233
   (d) 70,863
   (e) 600,000
   (f) 5,362,449

7. Thirty percent of 500 is equal to $\frac{30}{100} \times 5.0 \times$ __?__

8. The volume of the body of the sun has been estimated as about 337,000 million million cubic miles. Write the number of cubic miles in scientific notation.

3-2. Calculating with Large Numbers

Not only is scientific notation shorter in many cases but it makes certain calculations easier. We shall start with some rather simple ones. Suppose we want to find the value of the product: 100 × 1,000. The first factor is the product of two tens. The second is a product of three tens, so, we have $100 = 10^2$ and $1000 = 10^3$.

Then,

$$\begin{aligned} 100 \times 1000 &= 10^2 \times 10^3 \\ &= (10 \times 10) \times (10 \times 10 \times 10) \\ &= 10 \times 10 \times 10 \times 10 \times 10 \\ &= 10^5 \end{aligned}$$

Hence $10^2 \times 10^3 = 10^5$. Notice that the exponent 5 is the sum of the exponents, 2 and 3.

Let us look at another example: 1,000,000 × 100,000. Written in scientific notation this is $10^6 \times 10^5$. How many times does 10 appear as a factor in this product? Is $10^6 \times 10^5$ equal to 10^{11}?

This is one hundred billion, but it is simpler to leave the number in the form 10^{11} than to write a "1" followed by eleven zeros. Notice that again we added the exponents.

Suppose we wish to find the product of 93,000,000 and 10,000. In scientific notation, this would be:

$$\begin{aligned}(9.3 \times 10^7) \times 10^4 &= 9.3 \times (10^7 \times 10^4) \quad \text{By which property?}\\ &= 9.3 \times 10^{11}\end{aligned}$$

Now try a more difficult example:

$$\begin{aligned}93{,}000{,}000 \times 11{,}000 &= \\ &\left.\begin{aligned}&= (9.3 \times 10^7) \times (1.1 \times 10^4)\\ &= (9.3 \times 1.1) \times (10^7 \times 10^4)\end{aligned}\right\}\\ &= 10.23 \times 10^{11}\\ &= (1.023 \times 10) \times 10^{11}\\ &= 1.023 \times (10 \times 10^{11})\\ &= 1.023 \times 10^{12}\end{aligned}$$

Note: The order of the factors has been changed by using the associative and commutative properties of multiplication.

In studies of astronomy and space flight, especially, we encounter very large numbers. The planet Pluto has a mean distance from the sun of about 3666 million miles or 3.666×10^9 miles. Distances to the stars are usually measured in "light years." A light year is the distance that light travels in one year. This is a good way to measure such distances. If we expressed them in miles, the numbers would be so large that it would be difficult to write them, much less understand what they mean. But suppose we wish to estimate the number of miles there are in a light year. This will be done in the following class exercise.

Class Exercises 3-2

1. It has been determined that light travels about 186,000 miles per second. In parts (a) to (d) below, do not perform the multiplication, just indicate the product (an example of an indicated product is $2.4 \times 10 \times 56 \times 10^4$).

Using 186,000 miles per second as the speed of light,

(a) How far would light travel in 1 minute?

(b) How far would light travel in 1 hour?

(c) How far would light travel in 1 day?

(d) How far would light travel in 1 year?

(e) Find the number written in Part (d) and show that when "rounded" it is 5.9×10^{12}.

(f) The number written in Part (e) is about 6 ___?___.

(g) Why is the number written in Part (d) not the exact number of miles that light travels in one year? Try to give two reasons.

Exercises 3-2

1. Multiply, and express your answer in scientific notation:
   - (a) $6 \times 10^7 \times 10^3$
   - (b) $10^{13} \times 12 \times 10^5$
   - (c) $10^4 \times 3.5 \times 10^9$
   - (d) $300 \times 10^5 \times 20$
   - (e) $10^2 \times 10^5 \times 7.63$
   - (f) $60 \times 60 \times 60$
   - (g) $7 \times 3 \times 10^5$
   - (h) $9.3 \times 10^7 \times 10 \times 10^6$
2. Multiply, and write your answer in scientific notation:
   - (a) $9{,}000{,}000 \times 70{,}000$
   - (b) $125 \times 8{,}000{,}000$
   - (c) $25{,}000 \times 186{,}000$
   - (d) $1100 \times 5 \times 200{,}000$
3. Sound travels in air about one-fifth of a mile in one second. Answer the following questions, assuming that a space ship travels at a rate of speed five times the speed of sound. Indicate your answers in scientific notation.
   - (a) How far will sound travel in air in one day?
   - (b) How far will the space ship travel in 20 hours?
   - (c) How far will the space ship travel in 50 days?
   - (d) How far will the space ship travel in 2 years? Is this far enough to reach the sun?

4. The distance from the North Pole to the equator is about 10,000,000 meters.
   - (a) Express, in meters, the distance around the earth through the Poles in scientific notation.
   - (b) A meter is equal to one thousand millimeters. Express in scientific notation the distance in millimeters from the North Pole to the South Pole.
   - (c) One inch is about the same length as $2\frac{1}{2}$ centimeters. About how many centimeters will equal a distance of 40,000 feet?
5. The distance around the earth at the equator is about 25,000 miles. In one second, electricity travels a distance equal to about 8 times that around the earth at the equator. About how far will electricity travel in 10 hours?
6. Suppose you had the task of making ten million marks on paper and you made two marks each second. Could you have made 10,000,000 marks in one year? (One year is $60 \times 60 \times 24 \times 365$ seconds.)
7. The earth's speed in its orbit around the sun is a little less than seventy thousand miles per hour. About how far does the earth travel in its yearly journey around the sun?

3-3. Calculating with Small Numbers

We have been dealing almost entirely with large numbers. But we also deal with many very small numbers. The mass of the electron and the mass of the proton are typical of such very small quantities. How can we conveniently represent these exceptionally small quantities?

Suppose we start with a power of 10, say 10^4, and divide by 10. We get 10^3, for $\frac{10^4}{10} = \frac{10 \times 10 \times 10 \times 10}{10} = 10 \times 10 \times 10 = 10^3$.

Now divide 10^3 by 10, obtaining 10^2. Divide 10^2 by 10, obtaining the result 10. Now, starting with 10^4 and dividing by 10 three times, we obtained

$$10^4,\ 10^3,\ 10^2,\ 10^1.$$

Notice that the exponents decrease by one each time. Now divide 10^1 by 10. We know that the result is 1. Also, we see that if the exponents are to continue the pattern of decreasing by one at each stage, the next exponent should be 0. For this reason, it is convenient to define 10^0 as 1--that is, $10^0 = 1$. Now we have obtained the following:

$$\frac{10^4}{10} = 10^3,\quad \frac{10^3}{10} = 10^2,\quad \frac{10^2}{10} = 10^1,\quad \frac{10}{10} = 10^0, \text{ and } 10^0 = 1.$$

The exponent in each answer above is 1 less than the exponent immediately preceding it. This is reasonable since each time we divide by 10 we remove one factor of 10 from the numerator.

Again divide by 10: As the next number we get $\frac{10^0}{10} = \frac{1}{10}$. If the pattern of exponents is to continue, we should expect the next exponent to be 1 less than 0. This is the number which we write $^{-}1$; it is a negative number. Hence, it seems reasonable to define 10^{-1} as meaning $\frac{1}{10}$. Now divide 10^{-1} by 10. The number obtained is $\frac{1}{10} \div 10 = \frac{1}{10} \times \frac{1}{10} = \frac{1}{10^2}$. If the pattern of exponents is to continue, the new exponent should be 1 less than $^{-}1$, or $^{-}2$. Accordingly, we want to define 10^{-2} as meaning $\frac{1}{10^2}$. It is important to notice that the number 10^{-2} is not a negative number. It is a positive number, namely the number $\frac{1}{100}$.

For any positive integer n, then, we make the following definition of 10^{-n}.

Definition. *If* n *is a positive integer, we define*

10^n = (*the product of* n *tens*),

and

$10^{-n} = 1 \div 10^n = 1 \div$ (*the product of* n *tens*).

For n = 0, *we define*

$10^0 = 1$.

These definitions enable us to write powers of 10 as illustrated:

10^4	10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}
10,000	1000	100	10	1	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$

Each number indicated on the first line is equal to the number immediately below it.

Class Exercises 3-3

1. Express each of the following using negative exponents:

$$\frac{1}{10,000}, \quad \frac{1}{1,000,000}, \quad \frac{1}{10^7}, \quad \frac{1}{10^9}, \quad \frac{1}{100,000}.$$

2. Express each of the following in fractional form:

$$10^{-3}, \quad 10^{-5}, \quad 10^{-7}, \quad 10^{-6}.$$

3. Express each of the following using negative exponents:

Example: one-hundredth $= \frac{1}{100} = \frac{1}{10^2} = 10^{-2}$.

(a) one-thousandth
(b) one-millionth
(c) one-billionth
(d) one-trillionth

Recall now the meaning of a number written in decimal form. We know that $.4 = \frac{4}{10}$. Hence $.4 = \frac{4}{10} = 4 \times \frac{1}{10} = 4 \times 10^{-1}$. Also $.005 = \frac{5}{1000} = \frac{5}{10^3} = 5 \times \frac{1}{10^3} = 5 \times 10^{-3}$. This expresses each of these numbers in scientific notation.

Now consider $.42 = \frac{42}{100}$. This is true since $.42 = \frac{4}{10} + \frac{2}{100} = \frac{40}{100} + \frac{2}{100} = \frac{42}{100}$. To put this in scientific notation we write

$$.42 = \frac{42}{100} = \frac{4.2 \times 10}{10^2} = \frac{4.2}{10} = 4.2 \times \frac{1}{10} = 4.2 \times 10^{-1}.$$

Similarly,

$$0.000305 = \frac{305}{1,000,000} = \frac{3.05 \times 10^2}{10^6} = \frac{3.05}{10^4} = 3.05 \times 10^{-4}.$$

We can now write 0.16×10^{-4} in scientific notation:

$$0.16 \times 10^{-4} = 1.6 \times \frac{1}{10} \times \frac{1}{10^4} = 1.6 \times \frac{1}{10^5} = 1.6 \times 10^{-5}.$$

We first used scientific notation to represent very large numbers. We have just seen that the use of negative exponents makes it possible to express very small numbers also in scientific notation. Thus,

$$.0001 = 10^{-4}, \qquad .00000673 = 673 \times 10^{-8} = 6.73 \times 10^{-6},$$

and $\frac{376}{10^{12}} = 3.76 \times 10^{-10}$.

When a positive number less than 1 is written in scientific notation we see that the exponent is always a negative integer, for example, $.63 = 6.3 \times 10^{-1}$.

Now look back at the examples of scientific notation in Section 3-1. When 1, or any number between 1 and 10, is written in scientific notation the exponent is zero. Thus, $1 = 10^0$, $3 = 3 \times 10^0$, $6.79 = 6.79 \times 10^0$.

When a number greater than or equal to 10 is written in scientific notation the exponent will be a positive integer. Thus, $27 = 2.7 \times 10^1$, $10 = 10^1$, $4,680,000 = 4.68 \times 10^6$.

In summary, for a number written in scientific notation:

If $0 <$ *the number* < 1,
the exponent is a negative integer; e.g.:

$$.03 = 3 \times 10^{-2}.$$

If $1 <$ *the number* < 10,
the exponent is zero; e.g.:

$$3.4 = 3.4 \times 10^{0}.$$

If the number ≥ 10,
the exponent is a positive integer; e.g.:

$$137 = 1.37 \times 10^{2}.$$

Note that we are not now trying to represent negative numbers in scientific notation. After a little more work with negative numbers, you will see that it is easy enough to do.

Exercises 3-3

1. Write each of the following in scientific notation.
   (a) 0.093
   (b) 0.0001
   (c) $\frac{1}{10^6}$
   (d) 1
   (e) 0.00621
   (f) $\frac{1}{10^2}$
   (g) 0.7006
   (h) 0.000000907
   (i) 6
   (j) 0.0045
2. Write each of the following in decimal notation.
   (a) 9.3×10^{-5}
   (b) 1.07×10^{-1}
   (c) 10^{-6}
   (d) 5×10^{-4}
   (e) 7.065×10^{-3}
   (f) 10^{-1}
   (g) 14.3×10^{-7}
   (h) 385.76×10^{-6}

3. Write each of the following in scientific notation.

   (a) 63×10^4

   (b) 0.157×10^{-3}

   (c) 0.0000024

   (d) 5.265×10^{-5}

   (e) 362.35

   (f) $10^{-5} \times 432$

   (g) 0.00000000305

   (h) 69.5×10^{-1}

*4. Fill in the blanks in the following to make a true sentence. Notice that in some parts scientific notation is NOT used.

   (a) $0.006 = 6 \times 10^{\boxed{}}$

   (b) $0.000063 = \boxed{} \times 10^{-5}$

   (c) $0.0004015 = 4015 \times 10^{\boxed{}}$

   (d) $6000.0 = 0.06 \times 10^{\boxed{}}$

   (e) $0.213 = 2.13 \times 10^{\boxed{}}$

   (f) $0.213 = 213 \times 10^{\boxed{}}$

   (g) $0.213 = \boxed{} \times 10^{-5}$

   (h) $0.213 = \boxed{} \times 10^{4}$

*5. Can you think of the one non-negative number that we cannot express in scientific notation?

3-4. Multiplication: Large and Small Numbers

You have already multiplied numbers such as 10^3 and 10^5. Recall that $10^5 \times 10^3 = 10^{5+3} = 10^8$. Often, we need to multiply numbers where negative exponents appear in the scientific notation.

Multiply 4.3×10^{-5} by 2×10^{-3}

$$
\begin{aligned}
(4.3 \times 10^{-5}) \times (2 \times 10^{-3}) &= (4.3 \times 2) \times (10^{-5} \times 10^{-3}) \\
&= 8.6 \times \left(\frac{1}{10^5} \times \frac{1}{10^3}\right) = 8.6 \times \frac{1}{10^8} \\
&= 8.6 \times 10^{-8}.
\end{aligned}
$$

According to the above, $10^{{}^{-}5} \times 10^{{}^{-}3} = \frac{1}{10^5} \times \frac{1}{10^3} = \frac{1}{10^8} = 10^{{}^{-}8}$

What is the result of adding $^{-}5$ and $^{-}3$? Do you recall that $^{-}5 + {}^{-}3 = {}^{-}8$? So we have shown that

$$10^{{}^{-}5} \times 10^{{}^{-}3} = 10^{({}^{-}5 + {}^{-}3)}$$

Show as above that $10^{{}^{-}2} \times 10^{{}^{-}4} = 10^{({}^{-}2 + {}^{-}4)}$. Is the same procedure followed when the exponents are negative as when they are positive; that is, do you add the exponents in each instance? Be sure you see why the answer is yes.

Now multiply 4.3×10^5 by $2 \times 10^{{}^{-}3}$,

$$\begin{aligned}(4.3 \times 10^5) \times (2 \times 10^{{}^{-}3}) &= (4.3 \times 2) \times (10^5 \times 10^{{}^{-}3})\\ &= 8.6 \times (10^5 \times \frac{1}{10^3})\\ &= 8.6 \times \frac{10^5}{10^3}\\ &= 8.6 \times 10^2.\end{aligned}$$

It has been shown above that $10^5 \times 10^{{}^{-}3} = 10^2$. However $5 + {}^{-}3 = 2$ so that

$$10^5 \times 10^{{}^{-}3} = 10^{(5 + {}^{-}3)}.$$

Show as above that $10^{{}^{-}4} \times 10^3 = 10^{({}^{-}4 + 3)}$. We can now see that when we multiply 10^a by 10^b the result is $10^{(a + b)}$ no matter whether a and b are positive or negative. This illustrates the following

General Property. If a and b are any positive or negative integers then

$$10^a \times 10^b = 10^{a + b}.$$

Of course, the property is valid if either or both the exponents a and b are zero. For example $10^3 \times 10^0 = 10^3$, $10^0 \times 10^0 \times 10^0 = 1$.

There is another idea which is involved in some problems. This idea appears in the following where the answer is desired in scientific notation:

$$\begin{aligned}(4.7 \times 10^{-3}) \times (5.4 \times 10^{7}) &= (4.7 \times 5.4) \times (10^{-3} \times 10^{7}) \\ &= 25.38 \times 10^{(-3 + 7)} \\ &= 25.38 \times 10^{4} \\ &= (2.538 \times 10) \times 10^{4} \\ &= 2.538 \times (10 \times 10^{4}) \\ &= 2.538 \times 10^{5}\end{aligned}$$

In this problem the numbers 4.7 and 5.4 were multiplied to obtain 25.38 and then this number was written in scientific notation as 2.538 × 10. Then, 2.538 × 10 was multiplied by 10^4.

Exercises 3-4

1. Write the following products in scientific notation.
   - (a) $10^{5} \times 10^{2}$
   - (b) 0.3×10^{2}
   - (c) $10^{-7} \times 10^{-6}$
   - (d) 0.04×0.002
   - (e) 0.0001×0.007
   - (f) $(5.7 \times 10^{-3}) \times 10^{-7}$
   - (g) $10^{12} \times 10^{-3} \times 10^{15}$
   - (h) $10^{12} \times 10^{-7} \times 10^{-8}$
2. Write the following products in scientific notation.
   - (a) 0.0012×0.000024
   - (b) $6 \times 10^{-7} \times 9 \times 10^{-3}$
   - (c) $14 \times 10^{-3} \times 10^{-5}$
   - (d) $3 \times 10^{-6} \times 10^{-4}$
   - (e) $38 \times 10^{-3} \times 0.00012$
   - (f) 0.000896×0.00635
3. Using powers-of-ten notation find the products of the following:
   - (a) $10{,}000 \times 0.01$
   - (b) $0.00001 \times 10{,}000{,}000$
   - (c) $10^{17} \times 10^{-23}$
   - (d) $10^{6} \times \frac{1}{10^{4}} \times \frac{1}{10^{5}} \times 10^{-4}$
4. Multiply forty-nine thousandths by the number seven and six hundredths using scientific notation. Express your answer in scientific notation.

5. A large corporation decided to invest some of its surplus money in bonds. If 11 million dollars was invested at an average annual rate of $3\frac{3}{4}\%$, what was the annual income from this investment? Use scientific notation in the computation, and also express your answer in scientific notation.
6. On a certain date the debt of the U. S. Government, when rounded to the nearest 100 billion dollars, was 300 billion dollars. Assuming that the government pays an average rate of interest of 3.313%, what is the number of dollars interest paid each year? Express the answer in scientific notation.
7. If the mass of one atom of oxygen is 2.7×10^{-23} grams, what is the mass of 40×10^{27} atoms of oxygen? Express the answer in scientific notation.
8. A chemical mass unit, which is one-sixteenth the atomic mass of oxygen is approximately equal to 1.66×10^{-24} grams. What is the mass of a billion chemical mass units?

3-5. Division: Large and Small Numbers

The principles involved in division are suggested by those we have developed for multiplication. We saw how to divide 10^6 by 10^4, that is

$$\frac{10^6}{10^4} = \frac{10^4 \times 10^2}{10^4} = 10^2 = 10^{6-4}.$$

Another way to do this is to write

$$\frac{10^6}{10^4} = 10^6 \times \frac{1}{10^4} = 10^6 \times 10^{-4} = 10^{6+(^-4)} = 10^{6-4} = 10^2.$$

Also, from our definition of 10^{-n} as $\frac{1}{10^n}$ we see that

$$10^6 \div 10^{-4} = \frac{10^6}{10^{-4}} = \frac{10^6}{\frac{1}{10^4}} = \frac{10^6}{\frac{1}{10^4}} \times \frac{10^4}{10^4} = 10^6 \times 10^4 = 10^{10}.$$

But, recall from our study of subtraction for negative numbers in Chapter 1 that

$$6 - (^{-}4) = 6 + 4 = 10.$$

Thus, we see that

$$10^{6} \div 10^{-4} = \frac{10^{6}}{10^{-4}} = 10^{6 - (^{-}4)} = 10^{10}.$$

These examples suggest the following

General Property. *If* a *and* b *are any positive or negative integers then*

$$10^{a} \div 10^{b} = 10^{a - b}.$$

Let us illustrate the use of this property in the division of two very small numbers given in scientific notation. Suppose we wish to divide 8×10^{-3} by 2×10^{-7}.

$$\frac{8 \times 10^{-3}}{2 \times 10^{-7}} = \frac{8}{2} \times \frac{10^{-3}}{10^{-7}} = 4 \times 10^{-3 - (^{-}7)} = 4 \times 10^{4}.$$

Always remember, however, that we can check our work in such problems by dealing only with positive exponents, making use of the definition of 10^{-n}. Thus we may proceed as follows.

$$\frac{8 \times 10^{-3}}{2 \times 10^{-7}} = \frac{\frac{8}{10^{3}}}{\frac{2}{10^{7}}} = \frac{\frac{8}{10^{3}} \times 10^{7}}{\frac{2}{10^{7}} \times 10^{7}} = \frac{8 \times 10^{4}}{2}$$

$$= 4 \times 10^{4}.$$

Be sure to justify each of the operations in this development.

Class Exercises 3-5

1. Using the general property, perform the following divisions and check your work as we have done above:

   (a) $10^{-4} \div 10^{5}$ (b) $10^{-2} \div 10^{-7}$.

2. (a) Divide 10^7 by 10^2.
   (b) Why is $10^{(7-2)}$ equal to 10^5?
   (c) Are $10^7 \div 10^2$ and $10^{(7-2)}$ numerals for the same number or different numbers?
3. (a) Find $6 - {}^{-}3$.
   (b) Find $10^{(6 - {}^{-}3)}$.
   (c) Use the illustrative example above to determine whether $10^6 \div 10^{-3}$ and $10^{(6 - {}^{-}3)}$ are numerals for the same number.
4. (a) Find $10^{({}^{-}4 - 5)}$.
   (b) Is $10^{-4} \div 10^5$ equal to $10^{({}^{-}4 - 5)}$? Why? (You found the first number in 1a.)
5. Is $10^{-2} \div 10^{-7} = 10^{({}^{-}2 - {}^{-}7)}$?
6. Write another numeral for $10^m \div 10^n$.
7. Perform the indicated divisions.
   (a) $10^{11} \div 10^{-5}$ (b) $10^{-8} \div 10^{-9}$ (c) $10^{-3} \div 10^9$.
8. Is $(6 \times 10^5) \div (3 \times 10^2)$ equal to $\frac{6}{3} \times \frac{10^5}{10^2}$?
   Is the final answer 2×10^3?
9. Perform the indicated divisions and express the answer in scientific notation.
   (a) $(1.2 \times 10^{-4}) \div (4 \times 10^6)$
   (b) $(6.4 \times 10^{-6}) \div (3.2 \times 10^{-5})$
   (c) $(9 \times 10^4) \div (0.3 \times 10^{-2})$

Exercises 3-5

1. Write the answers to the following in scientific notation.

   (a) $10^5 \div 10^2$
   (b) $10^3 \div 10$
   (c) $10^{14} \div 10^4$
   (d) $10^{17} \div 10^{12}$
   (e) $10^{11} \div 10^{13}$
   (f) $10^{10} \div 10^{20}$
   (g) $10^6 \div 10^{12}$
   (h) $10^3 \div 10^4$

2. Write the answers to the following in scientific notation.

   (a) $10^5 \div 10^{-2}$
   (b) $10^3 \div 10^{-1}$
   (c) $10^{14} \div 10^{-4}$
   (d) $10^{17} \div 10^{-12}$
   (e) $10^{11} \div 10^{-13}$
   (f) $10^{10} \div 10^{-20}$
   (g) $10^6 \div 10^{-12}$
   (h) $10^3 \div 10^{-4}$

3. Write the answers to the following in scientific notation.

   (a) $10^{-5} \div 10^2$
   (b) $10^{-3} \div 10$
   (c) $10^{-14} \div 10^4$
   (d) $10^{-17} \div 10^{12}$
   (e) $10^{-11} \div 10^{13}$
   (f) $10^{-10} \div 10^{20}$
   (g) $10^{-6} \div 10^{12}$
   (h) $10^{-3} \div 10^4$

4. Write the answers to the following in scientific notation.

   (a) $10^{-5} \div 10^{-2}$
   (b) $10^{-14} \div 10^{-4}$
   (c) $10^{-11} \div 10^{-13}$
   (d) $10^{-6} \div 10^{-12}$
   (e) $10^{-3} \div 10^{-1}$
   (f) $10^{-17} \div 10^{-12}$
   (g) $10^{-10} \div 10^{-20}$
   (h) $10^{-3} \div 10^{-4}$

5. Write the answers to the following in scientific notation.

   (a) $(6 \times 10^{-5}) \div (3 \times 10^{-2})$
   (b) $(7 \times 10^{-3}) \div 10^4$
   (c) $(1.2 \times 10^6) \div 10^{-3}$
   (d) $(2.4 \times 10) \div 10^{-1}$
   (e) $\dfrac{9.6 \times 10^{-4}}{2.4 \times 10^{-2}}$
   (f) $\dfrac{7.6}{1.9 \times 10^3}$

6. Fill in the blank places with the proper symbol.

   (a) $12\% = \frac{12}{100} = \frac{12}{10^2} = 12 \times 10^{\square} = 1.2 \times 10^{\square}$

   (b) $46\% = \frac{46}{100} = \frac{46}{10^2} = 46 \times 10^{\square} = 4.6 \times 10^{\square}$

   (c) $0.3\% = \frac{0.3}{100} = \frac{0.3}{10^{\square}} = 0.3 \times 10^{\square} = 3 \times 10^{\square}$

   (d) $350\% = \frac{350}{100} = 350 \times 10^{\square} = 3.5 \times 10^{\square}$

   (e) $\frac{450}{3\%} = \frac{450}{3 \times 10^{\square}} = 150 \times 10^{\square} = 1.5 \times 10^{\square}$

   (f) $\frac{4800}{2.4\%} = \frac{4800}{2.4 \times 10^{\square}} = \frac{4.8 \times 10^{\square}}{2.4 \times 10^{\square}} = 2 \times 10^{\square}$

7. A city government has an income of $2,760,000 for this year. The income this year represents 3% of the total value of taxable property. What is the total value of taxable property? Use scientific notation in your computations.

8. A commuter pays $.40 per day for his fare. Would it be reasonable to expect that he will spend one million cents in fares before he retires? Assume that he travels 250 days per year.

*9. At the rate of ten dollars per second, about how many days would it take to spend a billion dollars? Assume this goes on 24 hours a day. (1 day $\approx 8.5 \times 10^4$ seconds)

*10. The tax raised in a certain county is $160,000 on an assessed valuation of $8,000,000. If Mr. Smith's tax is $400 what is the assessed value of his property?

*11. It costs about $35,000,000 to equip an armored division and about $14,000,000 to equip an infantry division. The cost of equipping an infantry division is what percent of the cost of equipping an armored division?

*12. The mass of the electron is approximately 9.11×10^{-28} grams and the mass of the proton is approximately 1.67×10^{-24} grams.

(a) Which is the greater?

(b) Approximately what is the ratio of the mass of the proton to the mass of the electron?

3-6. Use of Exponents in Multiplying and Dividing Decimals

You know how to multiply two numbers in decimal form and also how to divide one by another. When you find the product or quotient of two numbers, it is usually easy to decide where to place the decimal point. But, when a number of multiplications and divisions are required, you may have difficulty in deciding where the decimal point belongs in the final answer. By using powers-of-ten notation we can work entirely with whole numbers until the very end of a complex calculation and then fix the location of the decimal point in an easy way. Also, the exponent notation gives an easy way to explain our usual procedure for determining the decimal point location. We will illustrate these features in this section.

Suppose we wish to multiply 32.14 by 1.6. Where is the decimal point to be located in the product?

Of course, in such a simple example we see at a glance that the product must be a number greater than 32 but less than 64, and this tells us where to put the decimal point in our answer. If we use exponent notation we proceed as follows:

$$\begin{aligned} 32.14 \times 1.6 &= (3214 \times 10^{-2})(16 \times 10^{-1}) \\ &= (3214 \times 16)(10^{-2} \times 10^{-1} \\ &= (3214 \times 16)(10^{-3}) \\ &= 51424 \times 10^{-3} \\ &= 51.424 \end{aligned}$$

Here we multiply whole numbers only in the product (3214 × 16). The factor $10^{{}^{-}3}$ tells us where to place the decimal point. The factor $10^{{}^{-}3}$ tells us there should be 3 decimal places to the right of the decimal point in the product, or $51424 \times 10^{{}^{-}3} = 51.424$.

Also, from the way we arrived at ${}^{-}3$ we see the justification for the rule that the number of places to the right of the decimal point in the product will be the sum of the number of places to the right of the decimal point in the two factors of the product.

If you are in doubt about the location of the decimal point in a product or in a quotient, the exponent notation will make it easy for you to decide. The advantage here is that we deal only with whole numbers in our multiplication and then later worry about the position of the decimal point.

This form of exponent notation is similar to scientific notation but differs from it in that the first factor does not have to be a number less than 10.

Class Exercises 3-6

1. Use the above procedure to find each of the following products:

   (a) 6.14 × 0.42

   (b) 0.625 × 0.038

   (c) 649.3 × 14.68

   (d) 11.4 × 0.0031

We can use the same scheme in the division of decimals. As an example of how to proceed let us divide 14.72 by 6.1.

$$\frac{14.72}{6.1} = \frac{1472 \times 10^{{}^{-}2}}{61 \times 10^{{}^{-}1}} = \frac{1472}{61} \times \frac{10^{{}^{-}2}}{10^{{}^{-}1}}$$

$$= \frac{1472}{61} \times 10^{({}^{-}2 - {}^{-}1)} = \frac{1472}{61} \times 10^{{}^{-}1}$$

Now the division $\frac{1472}{61}$ is simply an operation with whole numbers and gives $\frac{1472}{61} = 24.13$, if we carry out the division, correct to the nearest hundredth.

Hence,

$$\frac{14.72}{6.1} = 24.13 \times 10^{-1} = 2.413,$$

correct to the nearest thousandth.

Here, also, we used powers of ten in such a fashion that we could divide one whole number by another in performing the actual division. The exponent $^{-}1$ simply fixed the position of the decimal point in the answer.

This notation is often a real advantage when doing more complicated problems requiring a number of operations with decimals. For example,

$$\frac{3015 \times .028}{.00007 \times .03 \times 1500} = \frac{(3015)(28)}{(7)(3)(15)} \times \frac{10^{-3}}{10^{-5} \times 10^{-2} \times 10^{2}}$$

$$= 268 \times 10^{2}$$

$$= 26800$$

To see how the answer was obtained, fill in all the steps of this calculation.

Exercises 3-6

1. Place the decimal point in the products to make the following number sentences true.

   (a) $6021 \times 0.00003 = (6021) \times (3 \times 10^{-5}) = 18063$

   (b) $3.42 \times 0.02 = (342 \times 10^{-2}) \times (2 \times 10^{-2}) = 684$

   (c) $2.5 \times 3,000 = (25 \times 10^{-1}) \times (3 \times 10^{3}) = 75$

   (d) $54.73 \times 7.3 = (5473 \times 10^{-2}) \times (73 \times 10^{-1}) = 399529$

   (e) $1200 \times 0.006 = (12 \times 10^{2}) \times (6 \times 10^{-3}) = 72$

2. Fill blanks with proper symbols.

   (a) $4.52 = 45.2 \times 10^{-1} = 452 \times 10^{\boxed{}}$

   (b) $0.012 = 1.2 \times 10^{-2} = 12 \times 10^{\boxed{}}$

   (c) $65000 = 6.5 \times 10^{\boxed{}} = 65 \times 10^{3}$

   (d) $38.216 = 382.16 \times 10^{-1} = 3821.6 \times 10^{-2} = 38216 \times 10^{\boxed{}}$

   (e) $6.37 \times 10^{4} = 63.7 \times 10^{3} = 637 \times 10^{2} = \boxed{} \times 10^{0}$

   (f) $0.003 \times 10^{5} = 3 \times 10^{\boxed{}} = 30 \times 10^{\boxed{}}$

   (g) $41.2 \times 10^{-3} = 0.412 \times 10^{-1} = \boxed{} \times 10^{0}$

3. Place the decimal point in the quotients to make the following sentences true.

   (a) $\frac{6004}{0.02} = \frac{6004 \times 10^{0}}{2 \times 10^{-2}} = 3002 \times 10^{2} = 3002$

   (b) $\frac{0.366}{0.06} = \frac{366 \times 10^{-3}}{6 \times 10^{-2}} = 61 \times 10^{-1} = 61$

   (c) $0.32\,\overline{)56.0064} = \frac{560064 \times 10^{-4}}{32 \times 10^{-2}} = 17502$

   (d) $\frac{0.084}{12000} = \frac{84 \times 10^{-3}}{12 \times 10^{3}} = 7$

   (e) $\frac{0.2}{0.00125} = \frac{2 \times 10^{-1}}{125 \times 10^{-5}} = 0.016 \times 10^{4} = 16$

4. Multiply using exponent notation.

   (a) 135×0.06

   (b) $76{,}000 \times 3{,}000 =$

   Hint: $(76 \times 10^{3}) \times (3 \times 10^{3})$

   (c) $18{,}000 \times 0.0003$

   (d) 0.0035×16.301

   (e) $6{,}000{,}000 \times 0.0275$

   (f) $0.07 \times 300 \times 0.02 \times 6{,}000$

5. Divide using exponent notation.

   (a) $6.3 \div 0.3$

   (b) $0.78 \div 13$

   (c) $\frac{8750}{8.75}$

   (d) $\frac{0.1470}{0.75}$

   (e) $0.27\,\overline{)0.84402}$

   (f) $1800\,\overline{)21.6}$

*6. Use exponents to place the decimal point in the answer.

$$\frac{418.6 \times 0.019}{0.13} = 6118$$

*7. How many pieces of popcorn each weighing 0.04 ounces will it take to make enough to fill 840 bags? Each bag will contain 6 ounces of popcorn.

8. BRAINBUSTER. A flying saucer can travel at 100,000 miles a second. About how long (in years) will it take it to visit and return from a star that is $5\frac{1}{3}$ light years away?

$$1 \text{ light year} \approx 5.9 \times 10^{12} \text{ miles}$$

$$1 \text{ year} \approx 3.2 \times 10^{7} \text{ seconds}$$

3-7. The Metric System; Metric Units of Length

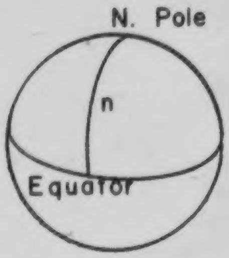
We have been using powers-of-ten notation in working with numbers, especially with very large and very small numbers. As an illustration of how powers of ten arise naturally in technical and scientific work we shall now study the metric system. As you will see, this system of measures is based upon powers of ten, and therefore the scientific notation we have studied is especially useful in dealing with metric quantities.

The system of measures used most widely in the United States is called the English system. Some of the units in this system are the inch, foot, yard, mile, gallon, pound, etc. As you have found, these units are a bit difficult to learn and remember because of the many different definitions required. In this English system you must learn that 12 inches = 1 foot, 3 feet = 1 yard, 5280 feet = 1 mile, 4 quarts = 1 gallon, 16 ounces = 1 pound. All the relations seem to use different numbers and there is no simple, easily remembered basic relationship among the units.

In most other countries, the system of measures is the metric system, in which the basic unit of linear measure is the meter. The metric system is a simplified system of weights and measures developed in 1789 by a group of French mathematicians. They decided that, since their system of numeration was a decimal (base 10) system it would be a good idea to have a decimal basis for a system of measures. In such a system the units of length would be some power of ten times a basic unit of length. Then it would be easy to convert from one unit to another. It would only require multiplying or dividing by a power of 10. We shall see that this makes it very much simpler to work with quantities expressed in metric units.

Metric Units of Length

The French mathematicians began by calculating the distance n from the North Pole to the equator on the meridian through Paris. For the basic unit of length they took $\frac{1}{10,000,000}$ of this distance. By defining the unit in this way the original distance could be measured again if the standard bar of unit length were ever lost.



$$1 \text{ Meter} = \frac{n}{10,000,000}$$

They named this new standard of length the meter and a standard meter bar was carefully preserved to assure uniformity in future meter units. This definition of the meter was used until October 15, 1960, when a new standard of the meter was agreed upon by delegates from 32 nations. This defines the meter in terms of the orange-red wave-lengths of krypton gas. Precisely, one meter is now defined as:

1 meter = 1,650,763.73 orange-red wave lengths
in a vacuum of an atom of the gas
krypton 86.

This new definition has the advantage that the unit is easily measured on an interferometer anywhere in the world. Also, it allows an accuracy of one part in one hundred million in linear measurements. Using the old standard bar of platinum-iridium an accuracy of one part in one million was the best obtainable.

In terms of our English system, a meter is a little longer than a yard, namely

1 meter = 39.37 in., (approx.)

Smaller units of length in the metric system are obtained by dividing by 10. Thus we define

$$1 \text{ decimeter} = \frac{1}{10} \text{ meter}$$

$$1 \text{ centimeter} = \frac{1}{10} \text{ decimeter} = \frac{1}{100} \text{ meter}$$

$$1 \text{ millimeter} = \frac{1}{10} \text{ centimeter} = \frac{1}{1000} \text{ meter.}$$

For longer units of length we simply multiply by 10. Thus, by definition,

1 dekameter = 10 meters

1 hectometer = 10 dekameters = 100 meters

1 kilometer = 10 hectometers = 1000 meters.

To emphasize the simplicity of the relations involved, we write these units in terms of meters, using scientific notation. Then the relations look like this.

1 millimeter = 10^{-3} meters

1 centimeter = 10^{-2} meters

1 decimeter = 10^{-1} meters

1 meter = 10^{0} meters

1 dekameter = 10^{1} meters

1 hectometer = 10^{2} meters

1 kilometer = 10^{3} meters

Many attempts have been made to get the United States to adopt the metric system for general use. Thomas Jefferson in the Continental Congress worked for a decimal system of money and measures but succeeded only in securing a decimal system of coinage. When John Quincy Adams was Secretary of State, he foresaw world metric standards in his 1821 "Report on Weights and Measures." In 1866, Congress authorized the use of the metric system, making it legal for those who wished to use it. Finally, in 1893, by act of Congress, the meter was made the standard of length in the United States. The yard and the pound are now officially defined in terms of the metric units, the meter and the kilogram.

A sudden change from our common units (yards, feet, inches, ounces, pounds) to metric units would undoubtedly cause confusion for a time. However, many people think that we will gradually change over to the metric system. Our scientists already use the metric system and people in most foreign countries use it also.

We summarize the definitions and abbreviations for the major metric units of length in Table A. In this table, notice how useful scientific notation is in showing relationships to the basic metric unit of length, the meter.

Table A

Linear Metric Units

Name of Unit	Abbreviation	Equivalent in Meters	Meter equivalent in Scientific Notation
1 millimeter	1 mm.	$\frac{1}{1000}$ m.	10^{-3} m.
1 centimeter	1 cm.	$\frac{1}{100}$ m.	10^{-2} m.
1 decimeter	1 dm.	$\frac{1}{10}$ m.	10^{-1} m.
1 meter	1 m.	1 m.	10^{0} m.
1 dekameter	1 dkm.	10 m.	10^{1} m.
1 hectometer	1 hm.	100 m	10^{2} m.
1 kilometer	1 km.	1000 m.	10^{3} m.

Notice that all the other metric units of length use the word "meter" with a prefix. These prefixes are also used to name other units of measure in the metric system.

Prefix	Meaning
milli	$\frac{1}{1000} = 10^{-3}$
centi	$\frac{1}{100} = 10^{-2}$
deci	$\frac{1}{10} = 10^{-1}$
deka	$10 = 10^{1}$
hecto	$100 = 10^{2}$
kilo	$1000 = 10^{3}$

In actual practice the hectometer, dekameter and decimeter are seldom used. The meter, centimeter, millimeter and kilometer are in very common usage and we shall devote most attention to them.

Two other prefixes which we should mention here are mega, meaning million and micro, meaning one-millionth. Thus

Prefix	Meaning
mega	$1{,}000{,}000 = 10^{6}$
micro	$\frac{1}{1{,}000{,}000} = 10^{-6}$

You often hear now of 3 megatons (3 million tons), 1 megacycle (1 million cycles). Even the slang term "megabuck" uses this classical Greek prefix!

In these days of nuclear and atomic studies very small quantities are often studied and lengths as small as one-millionth of a meter are common. The micron is defined as:

$$1 \text{ micron} = \text{one-millionth of a meter} = 10^{-6}\text{m.}$$

Thus 3 microns $= 3 \times 10^{-6}$m. $= 3 \times 10^{-3}$ millimeters, since 1 micron $= 10^{-3}$ millimeters. The usual symbol for micron is the Greek letter μ (read Mu). Hence $14\,\mu = 14$ microns $= 14 \times 10^{-6}$ m.

Have you encountered the terms megavolt, megohm, microwatt, microsecond, microfarad? If not, you will soon be discussing them in science class. Can you figure out what a micro-micron is?

Exercises 3-7a

1. Complete each of the following:

   (a) 1 kilometer = __________ hectometers

   (b) 1 kilometer = __________ dekameters

   (c) 1 kilometer = __________ meters

   (d) 1 kilometer = __________ decimeters

   (e) 1 kilometer = __________ centimeters

   (f) 1 kilometer = __________ millimeters

2. Complete each of the following:

   (a) 1111.111 meters = __________ kilometers

   (b) 5.342 meters = __________ centimeters

   (c) 245.36 meters = __________ kilometers

   (d) 0.564 m. = __________ mm.

   (e) 6043.278 m. = __________ km.

   (f) 2020.202 m. = __________ cm.

   (g) .015 mm. = __________ microns

3. Fill in each blank with the correct number.

   (a) 5 m. = __________ cm.

   (b) 200 cm. = __________ mm.

   (c) 500 m. = __________ km.

   (d) 2.54 cm. = __________ mm.

   (e) 1.5 km. = __________ m.

   (f) 3.25 m. = __________ cm.

   (g) 3500 m. = __________ km.

   (h) 474 cm. = __________ m.

   (i) 5.5 cm. = __________ mm.

   (j) 6.25 m. = __________ cm.

4. A meter was originally defined to be $\frac{1}{10,000,000}$ of the distance on the earth's surface from the North Pole to the equator. Assuming the earth is a sphere and using scientific notation, find the approximate number of

   (a) meters in the circumference of the earth;

   (b) kilometers in the circumference of the earth;

   (c) millimeters in the circumference of the earth.

*5. Express in scientific notation in terms of meters:

   (a) .013 millimeters

   (b) 2.34 centimeters

   (c) 6,730 kilometers

   (d) 694 microns

   (e) 1 mega-micron

Conversion to English Units

Since we use both English and metric units in this country, it is often necessary to convert from one system to another. The standard U.S. inch is now defined in terms of the metric standard by the relation

$$1 \text{ inch} = 2.54 \text{ cm.} \qquad \text{(definition of inch).}$$

As we have seen, this gives

$$39.37 \text{ in.} = 1 \text{ meter}, \quad \text{(approx.).}$$

In terms of yards, this says that

$$1 \text{ meter} = \frac{39.37}{36} \text{ yds. or}$$

$$1 \text{ meter} \approx 1.1 \text{ yds.}$$

For measurement of longer distances, it is often useful to convert miles to kilometers. From

$$1 \text{ meter} = 39.37 \text{ in.} = \frac{39.37}{12} \text{ ft.},$$

we see that

$$1 \text{ meter} = \frac{39.37}{12(5280)} \text{ miles.}$$

Hence,

$$1 \text{ kilometer} = \frac{(1000)(39.37)}{(12)(5280)} \text{ miles.}$$

Naturally, we now ask you to verify by actual calculation that this relation gives

$$1 \text{ kilometer} \approx 0.62 \text{ miles.}$$

Roughly speaking, then, 1 kilometer $\approx$ .6 of a mile or, for an even better approximation, 1 kilometer $\approx \frac{5}{8}$ of a mile.

Exercises 3-7b

1. Convert each of the following metric measurements to approximately equivalent English measurements.

   (a) 100 meters ≈ __________ yds.
   (b) 200 meters ≈ __________ yds.
   (c) 400 meters ≈ __________ yds.
   (d) 800 meters ≈ __________ yds.
   (e) 1500 meters ≈ __________ yds.
   (f) 1500 meters ≈ __________ miles

   A few of the standard distances for track and field events.

   (g) 10 kilometers ≈ __________ miles
   (h) 100 kilometers ≈ __________ miles

2. (a) The height of Mount Everest, world's highest mountain is 29,003 ft. Approximately what is this height, rounded to 29,000 ft., expressed in meters?

   (b) One of the greatest sea depths was measured as 34,219 ft. Express this depth in meters, approximately, after rounding to the nearest 100 ft.

3. (a) Show from our earlier relation between miles and kilometers that 1 mile $\approx$ 1.61 kilometers.

   (b) The mean distance from the earth to the sun is about 9.29×10^7 miles. About how many kilometers is this?

4. A common size of typewriting paper is $8\frac{1}{2}$ inches by 11 inches. What are these dimensions in centimeters?

5. (a) What is your height in centimeters?

   (b) In millimeters?

   (c) In microns?

6. Which is the faster speed, a speed of 100 ft. per second, or a speed of 3000 cm. per second?

3-8. Metric Units of Area

We have learned how to find the area of the interior of a simple closed curve for a variety of simple curves. We chose the area of a square region as the best unit to use in measuring the area of the interior of such a closed curve.

The metric unit for measuring areas is also a square region. We use as a basic unit the area of a square region with each edge of length one meter. The area of the interior of this square region is called one square meter (abbreviation sq.m.).

If you have a centimeter rule available you should draw a square of side 1 cm. in order to get some idea of the size of one square centimeter. From 1 m. = 100 cm. = 39.37 inches we see that

$$1 \text{ cm.} \approx .39 \text{ inch,}$$

or

$$1 \text{ cm.} \approx .4 \text{ inch.}$$

Compared to the basic unit of one square meter, the square centimeter is small indeed. Remember that 1 cm. = $\frac{1}{100}$ m., hence

$$1 \text{ sq. cm.} = \frac{1}{100} \times \frac{1}{100} \text{ sq.m.} = \frac{1}{10,000} \text{ sq.m.}$$

Here is another instance in which you may prefer to use exponent notation and write instead

$$1 \text{ sq. cm.} = 10^{-4} \text{ sq.m.}$$

Since the meter is the basic unit of length and the square meter the basic unit of area, it is important to give the various units of area in terms of square meters. The most commonly used units are listed in Table B.

Table B

Unit of Length	Unit of Area	Equivalent Area in Square Meters
millimeter	sq. millimeter	$\frac{1}{(1000)^2}$ sq. m.
centimeter	sq. centimeter	$\frac{1}{(100)^2}$ sq. m.
kilometer	sq. kilometer	$(1000)^2$ sq. m.

Here again we see that it is convenient to use exponent notation and write,

$$1 \text{ sq. mm.} = 10^{-6} \text{ sq. m.}$$

$$1 \text{ sq. cm.} = 10^{-4} \text{ sq. m.}$$

$$1 \text{ sq. km.} = 10^{6} \text{ sq. m.}$$

Exercises 3-8

1. Complete each of the following:
   Example: 1 sq. km. = 1000^2 sq. m. or 10^6 sq. m.
   (a) 1 sq. cm. = $(\frac{1}{100})^2$ sq. m. or __________ sq. m.
   (b) 1 sq. mm. = $(\frac{1}{1000})^2$ sq. m. or __________ sq. m.
   (c) 1 sq. cm. = 10^2 sq. mm. or __________ sq. mm.
   (d) 1 sq. m. = $(100)^2$ sq. cm. or __________ sq. cm.
   (e) 1 sq. m. = $(\frac{1}{1000})^2$ sq. km. or __________ sq. km.

2. Draw a sketch to illustrate (c) in Problem 1.

3. Find the area of a rectangular closed region with the following dimensions. Be sure that both dimensions are expressed in the same unit.

	Length	Width
(a)	35 cm.	9.2 cm.
(b)	1.68 m.	7.6 m.
(c)	.97 m.	37 cm.
(d)	1.25 mm.	1.2 cm.

4. Express the area in Problem 3 (b) in square centimeters.

5. What is the area of the interior of a circle whose radius is 6 m.? Use 3.14 for π and find the approximate number of square meters in the area.

6. Find the area in square meters of a square closed region 160 cm. on a side.

7. If the area of the interior of a parallelogram is 783 sq. cm. and the base is 27 cm., what is the altitude?

8. (a). What is the metric equivalent in sq. cm. of 1 sq. yd?
   1 yd. = 91.41 cm. Hence
   1 sq. yd $\approx (91.4)^2$ sq. cm, = ____?____ sq. cm.
   (b) Approximately how many sq. cm. equal 1 sq. inch?

9. (a) Verify that 1 sq. km. ≈ 0.386 sq. mile.

   (b) Verify that 1 sq. mile ≈ 2.59 sq. km.

10. (a) The land area of the earth is estimated to be about 149×10^6 sq. km. Approximately how many sq. miles is this?

    (b) The ocean area of the earth is estimated as 361×10^6 sq. km. Approximately how many sq. miles of ocean are there?

3-9. Metric Units of Volume

The metric unit for measuring volume is a cubical solid. The length of each edge of this cube is 1 meter. The volume of this cube is thus 1 cubic meter (abbreviation 1 cu. m.).

The cubic meter is a rather large unit of volume. A much smaller unit is the cubic centimeter (cu. cm.). As we have seen, the centimeter is about .4 in.; hence, the cu. cm. is a cube about the size of a small sugar cube. To suggest the size of the cubic meter, note that

$$1 \text{ cu. cm.} = \frac{1}{100} \times \frac{1}{100} \times \frac{1}{100} \text{ cu. m.} = \frac{1}{1,000,000} \text{ cu. m.}$$

In scientific notation,

$$1 \text{ cu. cm.} = 10^{-6} \text{ cu. m. and } 1 \text{ cu. m} = 10^{6} \text{cu. cm.}$$

By similar calculations we can find the commonly used multiples and subdivisions of the cubic meter. These are displayed in Table C.

Table C

Unit of Length	Unit of Volume	Equivalent Volume in Cubic Meters
millimeter	cu. mm.	$\frac{1}{(1000)^3}$ cu. m.
centimeter	cu. cm.	$\frac{1}{(100)^3}$ cu. m.
kilometer	cu. km.	$(1000)^3$ cu. m.

[sec. 3-9]

Here, even more than before, we see the convenience of the exponent notation in writing

$$1 \text{ cu. mm.} = 10^{-3} \times 10^{-3} \times 10^{-3} \text{ cu. m.} = 10^{-9} \text{ cu. m.}$$

$$1 \text{ cu. cm.} = 10^{-2} \times 10^{-2} \times 10^{-2} \text{ cu. m.} = 10^{-6} \text{ cu. m.}$$

$$1 \text{ cu. km.} = 10^{3} \times 10^{3} \times 10^{3} \text{ cu. m.} = 10^{9} \text{ cu. m.}$$

Look at the calculations above. You see that

$$10^3 \times 10^3 \times 10^3 = (10^3)^3 = 10^{3\cdot 3} = 10^9$$

$$10^{-2} \times 10^{-2} \times 10^{-2} = (10^{-2})^3 = 10^{(-2)\cdot 3} = 10^{-6}$$

$$10^{-3} \times 10^{-3} \times 10^{-3} = (10^{-3})^3 = 10^{(-3)\cdot 3} = 10^{-9}.$$

This illustrates a third

General Property: If a and b are any positive or negative integers, then

$$(10^a)^b = 10^{ab}.$$

Exercises 3-9

1. Complete each of the following:

   Example. There are $(1000)^3$ or 1,000,000,000 cu. m. in 1 cu. km.

   (a) There are 10^3 or ____________ cu. mm. in a cu. cm.

   (b) There are $(\frac{1}{100})^3$ or ____________ cu. m. in a cu. cm.

   (c) There are $(\frac{1}{1000})^3$ or ____________ cu. m. in a cu. mm.

   (d) There are $(10^6)^3$ or $10^{\square}$ ____________ cu. mm. in a cu. km.

2. A rectangular solid has dimensions of 6 cm., 7 cm., and 8.4 cm. Calculate the volume of the interior of this solid. Recall that the volume of the interior of a rectangular solid is equal to the product of the measures of the length, width, and height, when the measurements are expressed in the same unit.

3. What is the volume of the interior of a rectangular solid whose height is 14 mm. and whose base has an area of 36.5 sq. cm.?

3-10. Metric Units of Mass and Capacity

The metric unit for the measure of mass is defined as the mass of water contained by a vessel with a volume of one cubic centimeter. *The mass of one cubic centimeter of water is called a gram*. This is a very convenient definition, for when we know the volume of a container we immediately know the mass of water it can contain. For example, if the volume of the interior of a container is 500 cu. cm., then the mass of water it can contain is 500 grams. The important thing to note in this definition is that the numerical measures are the same.

When we speak of the volume of a box or other container, we frequently use the term *capacity*. By the capacity of a container we simply mean the total volume which the container will hold.

In talking about the volume of liquid a container will hold, we frequently use special units, such as, pint, quart, and gallon, in the English system. Thus, we may say the capacity of a tank is a certain number of gallons, and its volume is so many cubic feet.

In the metric system the usual unit of capacity is the *liter* (abbreviated ℓ.). One *liter* is defined as the capacity of a cubical box with edge of length 10 cm. (1 decimeter). Thus, one *liter* means a volume of 1000 cu. cm. We say a cube of edge 10 cm. has a volume of 1000 cu. cm. We say its capacity is one liter. It can contain a mass of 1000 grams (or one kilogram) of water.

One liter is approximately one quart, or more precisely,

1 liter = 1000 cu. cm. = 1.056712 qt.

The other most common metric measures of capacity are the

milliliter (mℓ.) = .001 liter

and the

kiloliter (kℓ.) = 1000 liters.

A mass of 1000 kilograms is called one *metric ton*. Hence, a metric ton contains 10^6 grams. The metric ton is the mass of 1 kiloliter of water.

The most used units of volume, capacity, and mass are summarized in Table D. Note especially that 1 *cu. cm. corresponds to* 1 *gm. of mass and to one milliliter of capacity*.

Table D

Unit of Volume	Unit of Mass	Unit of Capacity
1 cu. cm.	1 gm.	1 mℓ. = .001 ℓ.
1 cu. dm.	1 kgm.	1 ℓ.
(1000 cu. cm.)	(1000 gm.)	(1000 mℓ.)
1 cu. m.	1 metric ton	1 kℓ.
(1000 cu. dm)	(1000 kgm.)	(1000 liters)
(1,000,000 cu. cm.)	(1,000,000 gm.)	(1,000,000 mℓ.)

There are several abbreviations commonly used for the gram. The abbreviations, g. or gm., are both generally accepted and the form, gr., is also used. In this text we abbreviate gram as gm., kilogram as kgm., and milligram as mgm.

Exercises 3-10

1. The volume of a jar is 352.8 cu. cm. What is the mass of the water it can contain, expressed in:
   - (a) grams?
   - (b) kilograms?
2. (a) What is the capacity in milliliters of a rectangular tank of volume 673.5 cu. cm?
   - (b) What is its capacity in liters?
3. A cubical tank measures 6 feet 9 inches each way and is filled with water.
   - (a) Find its volume in cubic inches.
   - (b) Find its volume in cubic feet. Recall that 1728 cubic inches = 1 cubic foot.
   - (c) Find the weight of the water. Recall that 1 cubic foot of water weighs 62.4 lb.
4. The dimensions of the tank in Problem 3 are about 2.06 meters each way.
   - (a) Find its volume in cubic meters.
   - (b) Find its contents in liters. Recall that there are 1000 liters in a cubic meter.
   - (c) What is the mass of the water? Recall that 1 liter of water has mass of 1 kilogram.
5. How did the time needed to solve Problem 4 compare with the time to solve Problem 3? What is the main advantage of computing in the metric system?
6. A tank has a volume of 2500 cu. cm.
   - (a) What is the capacity of the tank in milliliters?
   - (b) How many kilograms of water will the tank hold?
   - (c) How many metric tons of water will the tank hold?

7. A cubical box has edges of length 30 cm.
   - (a) What is the volume of the box in cu. cm.?
   - (b) What is the capacity in liters?
   - (c) How many kilograms of water will the box hold? (Assume that it is watertight, of course!)

*8. The volume of the sun is estimated to be about 337,000 million million million cubic miles or

$$3.37 \times 10^{17} \text{ cu. miles.}$$

   - (a) Using the fact that 1 mile $\approx$ 1.6 kilometers, express the volume of the sun in cubic kilometers. (Simply indicate multiplications in your answer if you wish.)
   - (b) Express the sun's volume in cu. cm., leaving your answer in the form of indicated multiplication.

9. The British Imperial gallon, used in Canada and Great Britain, is equivalent to 1.20094 U.S. gallons, or,

$$1 \text{ British Imperial gal.} \approx 1.2 \text{ U.S. gal.}$$

   - (a) When you buy 5 "gallons" of gasoline in Canada, how many U.S. gallons do you receive?
   - (b) How many Imperial gallons are required to fill a barrel which holds 72 U.S. gallons?

Research Problems:

Use the Twentieth Yearbook of the National Council of Teachers of Mathematics as your reference book.

(a) Collect a list of all the various units of measurement for length, area, volume, and capacity that you can find listed in your reference book. Bring this list to school.

(b) Write a composition on "Why I Prefer the English System of Measures to the Metric System" or "Why I Prefer the Metric System of Measures to the English System." You may use your reference book to aid you in getting information. A good discussion is given in "The Metric System--Pro and Con," by Chauncey D. Leake and Ralph M. Drews; *Popular Mechanics*, December 1960.

(c) Which weighs more, a pound of feathers or a pound of gold?

Brief Summary of Relations among Units

The following table summarizes much of the work in the metric system. From it you can derive all the multiples and subdivisions of units of area, length, volume, mass, and capacity.

Table E

Length

10 millimeters (mm.) = 1 centimeter (cm.)

100 centimeters (cm.) = 1 meter (m.)

1000 meters (m.) = 1 kilometer (km.)

Capacity

1000 milliliters (mℓ.) = 1 liter (ℓ.) = 1000 cu. cm.

Mass

1000 milligrams (mgm.) = 1 gram (gm.)

1000 grams (gm.) = 1 kilogram (kgm.)

1000 kilograms (kgm.) = 1 metric ton

Some important conversion relations between corresponding English and metric units are listed below for reference.

Table F

Length

1 inch	=	2.54 cm.	(definition of the inch)
1 meter	≈	39.37 in.	
1 cm.	≈	.39 in.	
1 km.	≈	0.62 miles	
1 mile	≈	1.61 km.	

Capacity

1 liter	≈	1.0567 qt.
1 liter	≈	0.2642 gal.
1 gal	≈	3.785 liters

Chapter 4

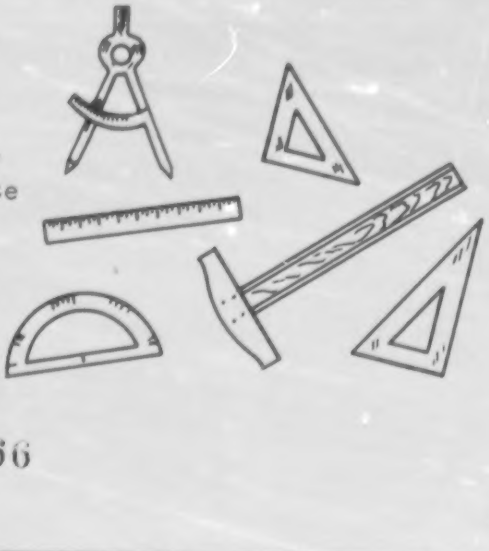
CONSTRUCTIONS, CONGRUENT TRIANGLES, AND THE PYTHAGOREAN PROPERTY

4-1. Introduction to Mathematical Drawings and Constructions

Drawing pictures and diagrams helps us solve many problems. Aeronautical engineers draw pictures of each part of a new aircraft to help them study the problems involved. Architects make drawings of floor plans and pictures of how completed buildings will look before the actual building is started. Theatrical directors sketch the stage and the location of properties to help decide how a certain scene should be staged. A carpenter makes drawings of the object that he is building. Electricians make diagrams to show how a machine should be wired. Many of your problems will be much easier to solve if you develop the habit of drawing pictures or diagrams to help you see the relations in your problem. Sometimes silly mistakes are made because students do not take the time to draw a picture of a problem situation.

For some problems a *rough sketch* of the situation is sufficient. Rough sketches can be drawn freehand in such cases. Although "roughly" drawn, the sketch may help you "see" the problem. There is no sense in wasting time on an accurate drawing if a rough sketch will serve.

Some problems can be solved by measuring drawings, but when drawings are used this way they should be accurate representations. Many tools are used to make accurate drawings. A man whose job is to make accurate drawings is called a *draftsman*. He uses a compass and a straightedge, but he uses many other tools. Draftsmen use such



tools as protractors, T-squares, 30-60 triangles, 45-45 triangles, rulers, parallel rulers, pantographs, and French curves to help make drawings accurate. You will be using some of these tools but others are too expensive to use at this time. Find out what the following tools are and how they are used by draftsmen:

(a) 30-60 triangle
(b) French curves
(c) Pantograph

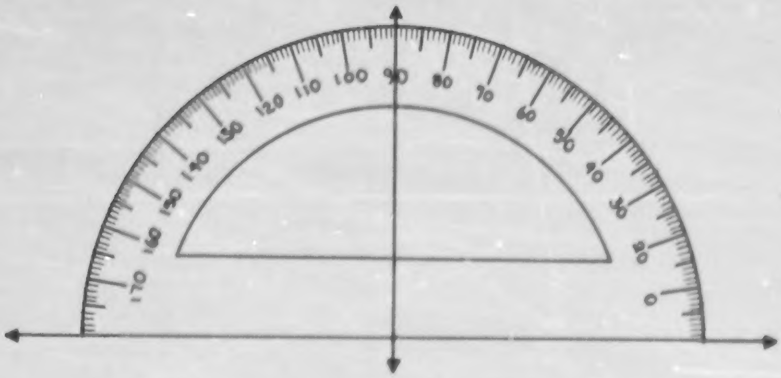
Tools alone cannot produce accuracy. You must use them in a way that gives accurate results. In the seventh grade you learned to use a protractor; however, some of you may need to review this with your teacher.

In measuring an angle, when a ray of an angle is not long enough to reach the scale of the protractor, extend the ray or lay a straightedge carefully along the ray. Check to see that you use the correct scale on the protractor.

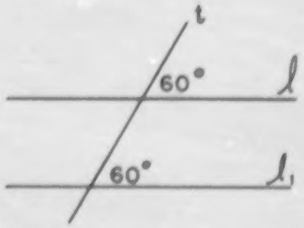
Plane figures, you recall, are figures that lie completely on a flat surface such as your paper or the chalkboard. Lines, angles, and polygons are examples of plane figures. Lines are the simplest of these figures. The relation of two or more lines on a plane is of special interest. Perpendicular lines are lines that intersect so as to form 90° angles. Parallel lines are two or more lines that do not intersect, or in set language, lines whose intersection is the empty set.

Perpendiculars may be drawn accurately with the aid of a protractor. At any point on a line, measure an angle of 90°. The rays may be extended to form lines. Lines, rays, or segments may be perpendicular to each other.

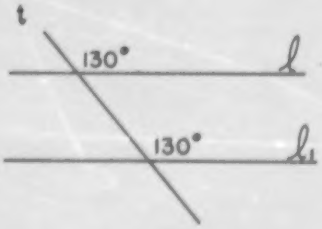
The edges of most rulers are considered to be parallel. A quick way to draw two parallel lines is simply to draw a line on each side of such a ruler without moving it. Of course, this method has limited use since all such pairs of parallel lines are the same distance apart. To overcome this difficulty the protractor is needed.



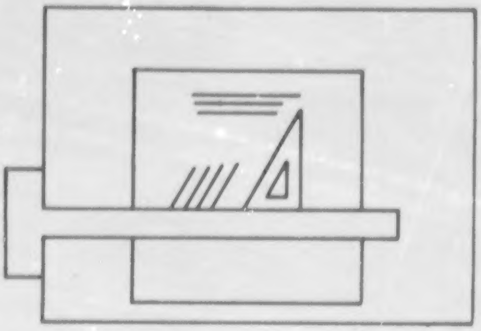
Protractors can be used to draw parallel lines accurately if you remember that corresponding angles are congruent when formed by parallel lines and a transversal. A transversal is a line that intersects two or more lines in distinct points.



To draw the figure at the right, use a straightedge to draw line ℓ . At any point on line ℓ draw a line t so that a 60° angle is formed. Draw a third line ℓ_1 , intersecting line t to form a 60° angle. Since the corresponding angles are congruent, line ℓ is parallel to line ℓ_1 .



Is line ℓ parallel to line ℓ_1 in each figure? How do you know?



T-squares and triangles also are useful in drawing parallel lines. The figure at the right illustrates the use of the triangle and the T-square in drawing lines that are parallel. By moving the triangle along the T-square or by moving the T-square, sets of parallel lines may be drawn.

Another type of accurate drawing that will be studied in this chapter is a compass and straightedge construction. These constructions are drawings that are made by using only two tools, a compass and a straightedge. Of course, a pencil (or chalk) can be used. Since "compass and straightedge construction" is such a long phrase, the single word construction will be used in this chapter to mean drawings that are made with a compass and straightedge only. A straightedge is a ruler without any units marked on it. It is used to make straight lines--not to measure their lengths.

Exercises 4-1

1. Draw an angle of 60° without using your protractor. Now measure the angle with your protractor. How close to 60° is your answer? Draw another to see if your estimate is improving.
2. Sketch the following angles with these measures in degrees.

   (a) 45 (c) 30 (e) 10 (g) 110
   (b) 90 (d) 120 (f) 160 (h) 65

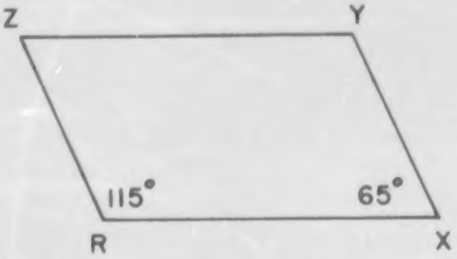
   Measure each with your protractor to see how well you estimated the size of each angle. Try again if you aren't satisfied with your first trial.
3. Which angles in problem 2 are acute angles? Which are obtuse angles?
4. With ruler and protractor draw a transversal intersecting one of two parallel lines at an angle of 80°
   (a) How many pairs of corresponding angles can you find?
   (b) How many pairs of vertical angles are there in your drawing?
   (c) How many degrees are there in each pair of vertical angles? in each pair of corresponding angles?
   (d) Can you draw three or more parallel lines each intersecting a transversal at an angle of 80°?

5. With ruler and protractor, draw a triangle that has two angles of 60° with a side $2\frac{1}{2}$ inches long between the vertices of the angles.

6. Using your protractor and ruler, draw a triangle with sides $6\frac{1}{2}$ cm. and $4\frac{4}{5}$ cm. The angle formed by these sides is 110°. What kind of triangle is this?

7. Do you recall that a parallelogram is any quadrilateral whose opposite sides lie on parallel lines? Is a square a parallelogram? Are all rectangles parallelograms?
   (a) Draw rectangle ABCD with ruler and protractor. What size is the angle at vertex A? at vertex B?
   (b) Are A and B consecutive vertices of parallelogram ABCD?
   (c) Recall a property of parallelograms: The angles of a parallelogram at two consecutive vertices are supplementary. The sum of the measures of angle A and angle B is ______. State the sum of the measures of angle B and angle C. Do the same with each pair of consecutive angles.

8. In parallelogram RXYZ



$m(\angle X) + m(\angle Y) =$ ______.

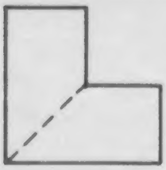
$m(\angle Z) + m(\angle R) =$ ______.

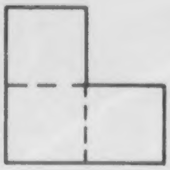
$m(\angle Y) + m(\angle Z) =$ ______.

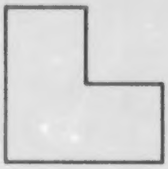
$m(\angle R) + m(\angle X) =$ ______.

9. Using your ruler and protractor, draw a parallelogram with sides of 1 inch and 2 inches and with one angle of 75°.

10. BRAINBUSTER. A farmer plans to divide his land equally among his sons. The land is shaped like the diagram. When he had two sons, he planned to divide it as shown. It was easy to divide when he had three sons. Now he has four sons. How can he divide the land so that each son gets the same amount in the same shape?


Divided for 2 sons


Divided for 3 sons


How should it be divided for 4 sons?

4-2. Basic Constructions

In drawing geometric figures, you have used several mechanical aids so that you could draw these pictures accurately. In the 2000-year period since the development of geometry as a science by Euclid and other Greek scholars, to place certain restrictions on the tools that could be used has been considered useful by many writers of geometry texts. Under these restrictions, a straightedge may be used to make straight line segments. A compass may be used to draw circles and arcs. An arc is any connected portion of a circle.

In your constructions you may use a ruler as a straightedge. In geometry, however, we think of a straightedge as having no marks on it with which measurements may be made.

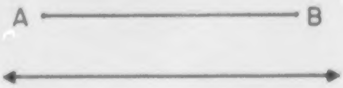
You may observe that we draw, or construct, a line segment by tracing with a pencil along an instrument which has a straight edge. On the other hand, our method of drawing a circle is quite different. In constructions we use a special instrument--the compass. We do not trace with a pencil along an object which has a circular-shaped edge. Efforts to devise an instrument with which a straight line segment could be drawn without using a straightedge, form an interesting chapter in the history of geometry. These efforts were successful only about 100 years ago.

Geometric drawings made with compass and straightedge only are called constructions. In this section you will learn several basic constructions used in geometry.

Follow the directions and complete each construction 1 through 4.

1. Copying a segment

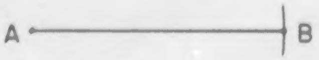
(a) Draw a working line a little longer than the segment that is to be copied.



Step a

(b) Place the point of the compass at one endpoint of the segment that is to be copied.

(c) Open the compass until the pencil touches the other endpoint. (The distance between the point of the compass and the end of the pencil is the length of the radius of the compass.)



Steps b and c

(d) Without changing the radius of the compass, place the point of the compass on the working line at A' (any point) and mark an arc where the pencil crosses the line. The segment from the point A' where the compass point is placed, to B' (the intersection of the arc and the line) is the same length as the original segment.

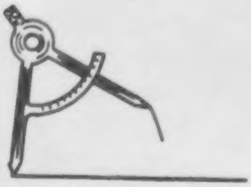
Step d

Any time that you need segments of equal length, this construction is used.

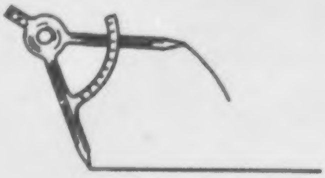
2. Bisecting a line segment

The word *bisect* means to divide into two equal parts.

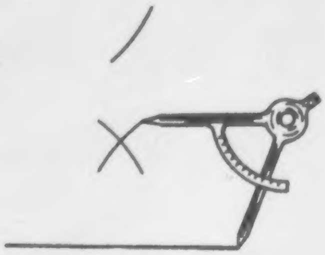
(a) Place the point of the compass at one endpoint of the segment. Set the compass so that its radius is more than half of the distance between the two endpoints.



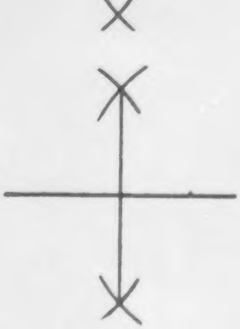
(b) Draw arcs above and below the center of the segment. Be sure the arcs are long enough to include points above and below the center.



(c) Without changing the radius of the compass, place the compass point at the other endpoint. Draw arcs that cross the first two arcs.



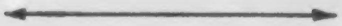
(d) Draw a line through the points where the arcs cross. This line bisects the original segment.



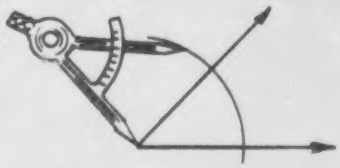
Measure the two parts obtained in (d). Are they the same length? What is the relation between the segment and the bisector that has been constructed?

3. Copying an angle

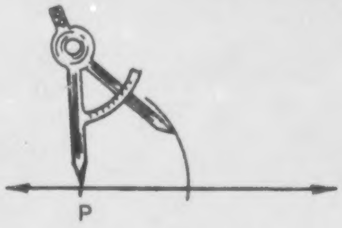
(a) Draw a base line, part of which will be used as one ray of the angle.



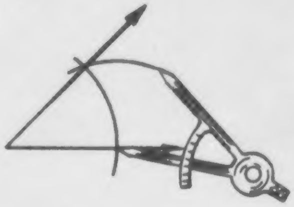
(b) Place the point of the compass at the vertex of the angle and draw an arc through both rays of the angle.



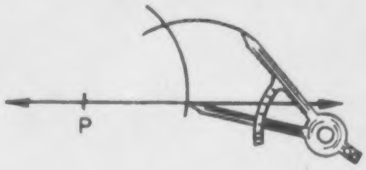
(c) From a point P, on the base line, draw an arc with the radius used in (b).



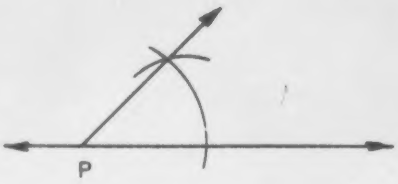
(d) Place the point of the compass at the intersection of one ray of the original angle and the arc that crosses it. Place the pencil of the compass on the other intersection.



(e) With the compass set as determined in (d), place the point at the intersection of the base line and the arc. Draw an arc that crosses the arc drawn in (c).



(f) Draw a ray from P on the base line through the intersection of the arcs.



Use your protractor to check this construction.

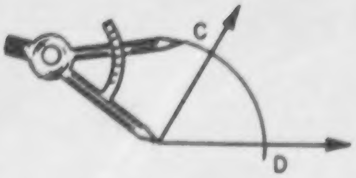
4. Bisecting an angle

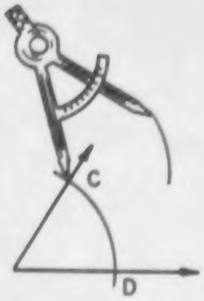
The four figures illustrate the steps in bisecting an angle with straightedge and compass.

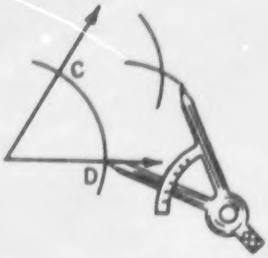
Study these steps in the constructions. Then draw an angle on a piece of paper and bisect it.

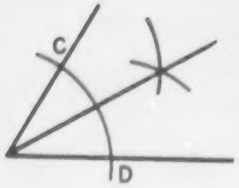
Can you state in your own words what is to be done in each step?

Use your protractor to measure the two angles that you have constructed. Are they equal in measure?









Exercises 4-2a

1. Use your ruler to draw a horizontal line $1\frac{1}{2}$ inches long. Construct on a given vertical line a segment of the same length.

2. Use your ruler to draw a vertical segment $2\frac{1}{8}$ inches long. Construct on a given horizontal line a segment of the same length.

3. Use your ruler to draw an oblique segment 5 centimeters long. Construct on a given horizontal line a segment of the same length.

4. Bisect each segment that you constructed in Problems 1 through 3. Use only compass and straightedge.

5. Draw an acute angle and an obtuse angle. Copy each angle using straightedge and compass only.

6. Draw an acute angle and an obtuse angle. Bisect each angle using straightedge and compass only.

7. (a) Draw a large triangle. Bisect each angle of the triangle. Extend the bisectors until they intersect.
   (b) When three or more lines intersect in one point, they are called concurrent lines. Do the bisectors appear to be concurrent lines?

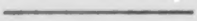
8. Draw a segment and then divide it into 4 equal parts. Use straightedge and compass only.

9. Draw an obtuse angle and construct rays which divide the angle into four congruent angles.

The basic constructions that you have studied can be used in many different ways. At this time you will explore only a few. When you study geometry in high school, you will find many more. This will be a discovery lesson. If necessary, short explanations will accompany a problem.

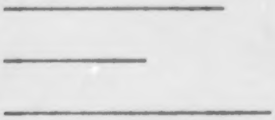
Exercises 4-2b

1. Draw a line segment about as long as this one.



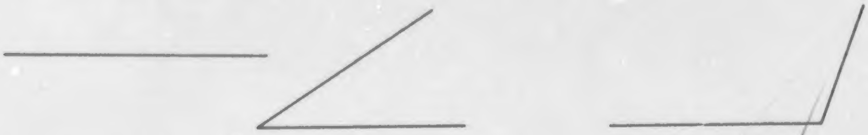
   (a) Using this length as a radius, draw a circle with one endpoint of the segment as the center.
   (b) With the same radius and the other endpoint as center, draw another circle.
   (c) At how many points do the circles intersect?
   (d) Choose one point where the two circles intersect and draw segments from this point to each endpoint of the original segment.
   (e) Compare the measures of the three segments.
   (f) What kind of triangle did you construct?

2. Construct a triangle whose sides are the same lengths as the segments given here.



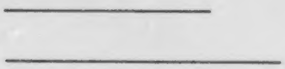
   You can use the plan in Problem 1, except that circles in steps (a) and (b) will not have the same radius.

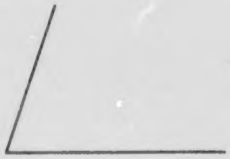
3. (a) Construct a triangle with a base the same length as the segment drawn here, and with the angles at each end of the base congruent to these angles.



   Hint: Use the base as one side of each angle.
   (b) Will all triangles constructed with these measures look alike? This construction is used to draw triangles when two angles and the side common to these angles are known.

4. (a) Construct a triangle with two sides the same size as these segments and with the angle formed by these segments the same size as the angle drawn here.





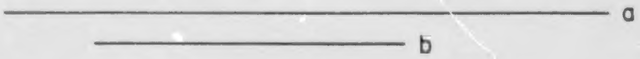
   (b) Will all triangles constructed with these measures look alike? This construction is used when two sides and the angle determined by those sides are known.

5. Construct a right triangle that has one acute angle of 60°. Hint: Can an equilateral triangle be used as the basis for this? How many degrees are there in each angle of an equilateral triangle? Two right triangles can be made from an equilateral triangle in two ways, using construction 4 or using construction 2. Try both methods and check your construction with a protractor.

6. Draw the following:
   (a) 3 concurrent lines
   (b) 4 concurrent lines
   (c) 5 concurrent lines

7. (a) Draw three rays such that the endpoints of the rays are the only point of intersection.
   (b) How many angles are formed by the rays in part (a)?

8. Construct a segment that has the same length as the difference between the lengths of these segments.



9. (a) Draw a triangle, then find, by construction, the midpoint of each side. Connect each of these midpoints to the opposite vertex. These segments are called the medians of a triangle.
   (b) Are the medians concurrent?

A polygon is a simple closed curve formed by straight line segments. A polygon with four sides is a quadrilateral. There are many kinds of quadrilaterals. Trapezoids, parallelograms, rectangles, and squares are all quadrilaterals. A polygon with five sides is a pentagon, a six-sided polygon is a hexagon, and an eight-sided polygon is an octagon. If the sides are all the same length and the angles all have the same measure, it is a regular polygon.

If every vertex of a polygon is a point on a circle, we say that the polygon is inscribed in the circle. In this section, you will use the constructions you have learned in order to inscribe equilateral triangles, squares, hexagons, and octagons. The problems contain enough clues for you to do all of these constructions.

Exercises 4-2c

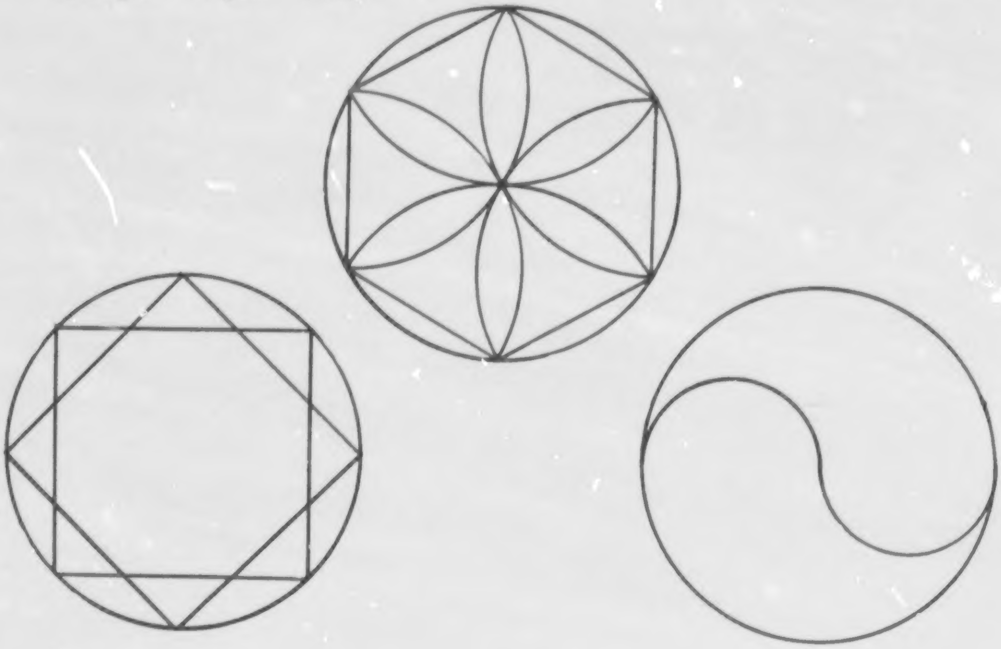
1. Draw a circle of radius 2 inches. With a compass set to the radius of the circle and starting with any point on the circle, mark off an arc on the circle. Move the point of the compass to one point where that arc crosses the circle. Mark another arc on the circle. Continue until the arc drawn falls at the starting point. If you do this carefully you will discover that the last arc drawn falls exactly on the starting point.
   - (a) How many arcs are there?
   - (b) Connect each intersection of the circle and an arc to the intersection on each side of it.
   - (c) What figure do these segments form?
   - (d) How can you use these points to construct an equilateral triangle?
   - (e) How can you form a six-pointed star?

2. Draw a circle and one diameter. Use your protractor to draw a diameter perpendicular to the first diameter. Connect the endpoints of the diameters in order.
   (a) What figure does this form?
   (b) How can you form a polygon with twice as many sides? There are two ways that this can be done. Can you find both of them?

   (Since you used your protractor, the figure you have drawn is not called a construction. We will learn in Section 4-5 how to construct a right angle.)

3. Construct a circle and construct in the circle an inscribed equilateral triangle, an inscribed hexagon, and an inscribed polygon of 12 sides.

4. Many designs can be formed with these basic constructions. Three are given here. See if you can copy them; then make up designs of your own.



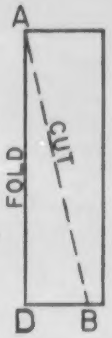
4-3. Symmetry

In the last section you worked with geometric constructions. In this section you will explore some of the properties of the figures you constructed. Most of the constructions are examples of symmetry and also examples of congruence.

This section is developed so that you will be able to discover for yourself what is meant by symmetry, and in Sections 4-4 and 4-5 you will study congruence.

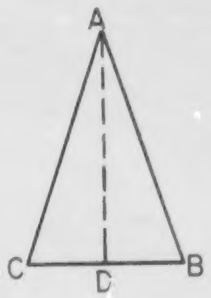
Class Exercises 4-3

1. (a) Fold a sheet of notebook paper (or other paper with square corners) down the middle. Starting at the folded edge, cut or tear off a right triangle with the longer leg along the fold, as in the second figure. Cut a right triangle from both sides of the folded paper at the same time. The right-hand figure above is a "double" sheet. Unfold the part which remains. What shape is it?



(b) Label as A the vertex at the fold, and the other vertices as B and C. Label as D the intersection of the fold and the side. The figure now resembles the figure below.

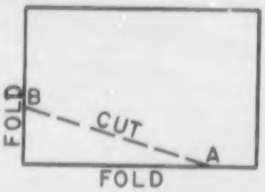
(c) Refold your triangle along AD. Do right triangles ABD and ACD exactly fit over each other? We say that triangle ABC has symmetry with respect to the line $\overleftrightarrow{AD}$ because when folded along $\overline{AD}$ the two halves exactly fit. Line $\overleftrightarrow{AD}$ is an axis of symmetry of the triangle.



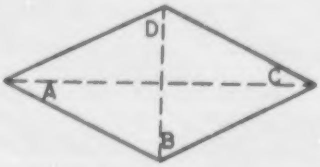
(d) How many axes of symmetry does an isosceles triangle have? An equilateral triangle? A scalene triangle?

2. (a) Take another piece of notebook paper and fold it lengthwise down the middle.

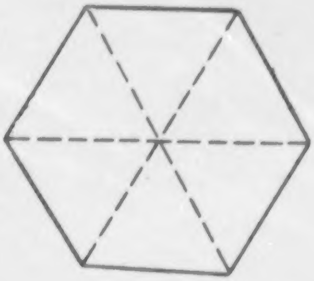
(b) Take your folded sheet and fold it crosswise down the middle (so that the crease folds along itself). Cut off the corner where the folds meet, as indicated in the figure at the right. Unfold the piece you cut off. What shape is it?



(c) Label your figure as in the figure at the right. If you fold along $\overline{AC}$, do the two halves exactly fit? What happens if you fold along $\overline{DB}$? Is there an axis of symmetry? How many?

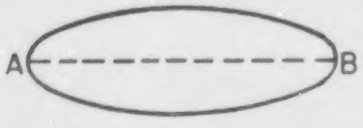


3. Look at the regular hexagon in this figure. Does each dotted line determine an axis of symmetry? There are other axes of symmetry. Can you find them? How many axes of symmetry does a regular hexagon have?



4. Draw a circle and one of its diameters. Is this diameter on an axis of symmetry? Does a circle have other axes of symmetry? Are there 5 axes of symmetry? 100? 10^5? Are there more than any number you may name?

5. Look at the ellipse in the figure to the right. It is a figure you get if you slice off the tip of a cone but do not slice straight across. Is $\overleftrightarrow{AB}$ an axis of symmetry? Are there others? How many axes of symmetry does an ellipse have? The segment $\overline{AB}$ is called the major axis of the ellipse. On another axis of symmetry is a minor axis of the ellipse. Why do you think $\overline{AB}$ is called the major axis? Where is the minor axis?



From these exercises you have learned that many of the geometric figures you know are symmetrical with respect to a line. Many ornamental designs and decorations also have such symmetry.

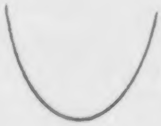
Definition. A figure is symmetrical with respect to a line ℓ, if for each point A on the figure, there is a point B on the figure for which ℓ is a bisector of $\overline{AB}$ and is perpendicular to $\overline{AB}$.

Exercises 4-3

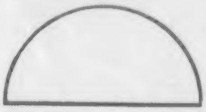
1. Draw a rectangle and draw its axes of symmetry. Label each axis of symmetry. How many axes of symmetry does a rectangle have?
2. Draw an equilateral triangle, and label each axis of symmetry. How many axes of symmetry are there?
3. Draw a square, and label each axis of symmetry. How many axes of symmetry does a square have?

4. Draw and label the axes of symmetry, if there are any, for each of the figures. How many axes of symmetry does each figure have?

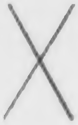
(a)



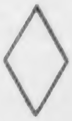
(b)



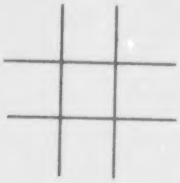
(c)



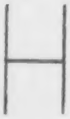
(d)



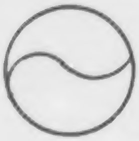
(e)



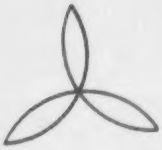
(f)



(g)



(h)

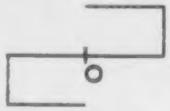


5. Fold a piece of paper down the middle and then cut designs in it. Unfold. Is the design symmetrical with respect to the fold? Is the fold an axis of symmetry?

6. Fold a piece of paper down the middle, and then fold it again in the middle, perpendicular to the first fold. Cut a design in it and unfold. Where are the axes of symmetry?

7. We say that a circle is symmetrical with respect to a point, its center, and that an ellipse is symmetrical with respect a point, its center (the point where its major and minor axes intersect). We also say that the figure below is symmetrical

with respect to the point O. Describe in your own words what you think is meant by symmetry with respect to a point.



Which of the figures in Problem 4 have symmetry with respect to a point?

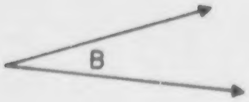
8. When an orange is cut through the center in such a way that each section of the orange is cut in half, we may think of the surfaces made by the cut as symmetrical. Symmetry of this kind is symmetry with respect to a plane. Name other objects that are symmetrical with respect to a plane.

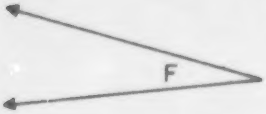
4-4. Congruent Triangles

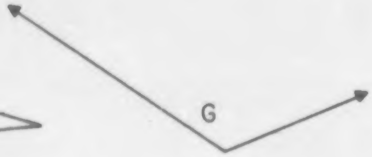
In Class Exercises 4-3, Problem 1, by paper folding and cutting, you produced an isosceles triangle. The axis of symmetry (the fold) divides the isosceles triangle into two right triangles which have the same size and shape. When two figures have the same size and shape we say that they are congruent. The two right triangles are congruent triangles.

Can you think of other congruent figures? Are two circles congruent, if each has a radius of five inches? Are two line segments having the same length congruent?

Since an angle is a geometric figure, may we talk about two congruent angles? Congruent figures have the same size and shape. Now if two angles have the same measure they have the same size, and to say that two angles have the same size means that they have the same measure. Appearance tells us that two angles with equal measures have the same shape.





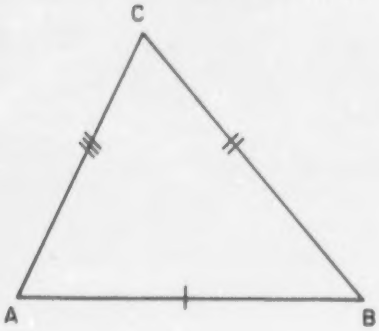


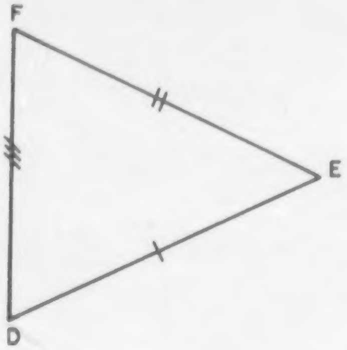
Angles B and F, as shown, have equal measures. We may say $\angle B$ is congruent to $\angle F$, and we may write $\angle B \cong \angle F$, where the symbol " $\cong$ " stands for the phrase "is congruent to." Is $\angle F \cong \angle G$?

We know that two circles are congruent if they have the same radius. Two squares are congruent if they both have the same measure for their sides. Two line segments are congruent if they both have the same length. Two angles are congruent if their measures are the same.

Are two rectangles congruent if their bases are equal? No. If their bases and heights are equal? Yes. You can see that the rectangle requires two conditions for congruency.

Triangles are so basic in much of mathematics, science, and engineering, that we need to know conditions under which triangles are congruent. The situation here involves more conditions than in the figures we have already discussed.





If triangle DEF were traced on paper and the paper cut along the sides of the triangle, the paper model would represent a triangle and its interior. The paper model could be placed on triangle ABC and the two triangles would exactly fit. The two triangles are congruent. If point D were placed on point A with $\overline{DF}$ along $\overline{AC}$, point F would fall on point C, and point E would fall on point B. In these two triangles there would be these pairs of congruent segments and congruent angles:

$\overline{AB} \cong \overline{DE}$ $\angle B \cong \angle E$

$\overline{CB} \cong \overline{FE}$ $\angle C \cong \angle F$

$\overline{CA} \cong \overline{FD}$ $\angle A \cong \angle D$

Use your ruler and protractor to check these measures.

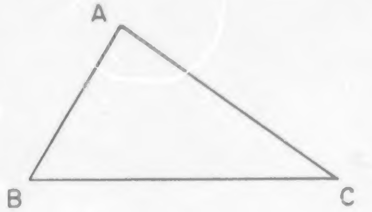
Recall that another way of expressing $\angle B \cong \angle E$ is $m(\angle B) = m(\angle E)$. Our choice of expression will depend upon whether we wish to emphasize the angles as being congruent figures or the measures as being equal numbers.

If two triangles are congruent, then for each angle or side of one triangle there is a congruent angle or side in the other triangle.

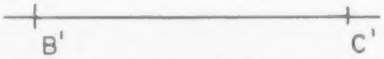
In the Class Exercises, you will study with your teacher conditions which will make two triangles congruent.

Class Exercises 4-4

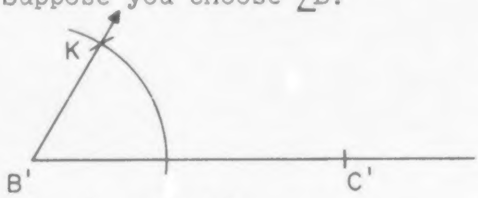
1. Construct a triangle which is congruent to triangle ABC.



You might start this construction by constructing a segment $\overline{B'C'}$ congruent to $\overline{BC}$.



Next you might think of constructing an angle congruent to either $\angle B$ or $\angle C$. Suppose you choose $\angle B$.

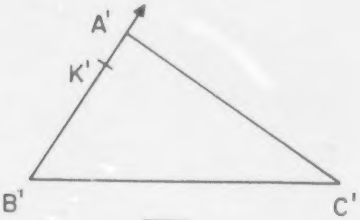


The figure shows the construction of ray $\overrightarrow{B'K}$ so that $\angle B' \cong \angle B$.

As a next step two possibilities might be considered:

(1) Mark A' on $\overrightarrow{B'K}$ so that

$\overline{B'A'} \cong \overline{BA}$

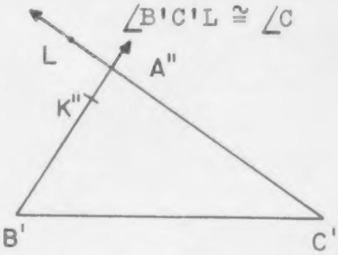


Draw $\overline{A'C'}$

or

(2) Construct ray $\overrightarrow{C'L}$ so that

$\angle B'C'L \cong \angle C$



Mark A".

You now have a triangle A'B'C' and a triangle A"B'C'. In both cases only three parts of triangle ABC have been copied.

Is triangle A'B'C' $\cong$ ABC? Is triangle A"B'C' $\cong$ ABC? In order to obtain the answers to these questions you must measure the parts of triangle A'B'C' and of triangle A"B'C' to see if these triangles are congruent to ABC.

If your constructions and measurements are correct you will find that

$$\Delta ABC \cong \Delta A'B'C' \quad \text{and} \quad \Delta ABC \cong \Delta A''B'C'.$$

In (1) by copying two sides and the included angle of triangle ABC, you have been able to construct a triangle congruent to triangle ABC. In (2) by copying two angles and the included side, you have been able to construct a triangle congruent to triangle ABC.

In (1) we refer to the included angle and in (2) to the included side. Do you see why the word "included" is convenient? In the first case the sides are part of the angle and in the second case the side is part of two angles.

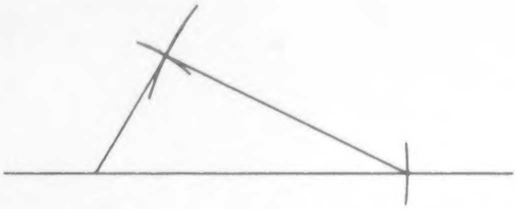
While one construction is not sufficient evidence on which to base a conclusion, your experience with that of your teacher and your classmates should convince you of the following properties of two congruent triangles.

Two triangles are congruent if two sides and the included angle of one triangle are congruent respectively to two sides and the included angle of the other triangle. We will refer to this property as Property S.A.S. (Side, Angle, Side).

Two triangles are congruent if two angles and the included side of one triangle are congruent respectively to two angles and the included side of the other triangle. We will refer to this property as Property A.S.A. (Angle, Side, Angle).

You are asked to accept these two properties and a third one to be developed in Problem 2, on the basis of experiment. As you continue you will use these three properties as a means of showing other properties.

2. Construct a triangle using $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ as sides. Your work should look like this figure:



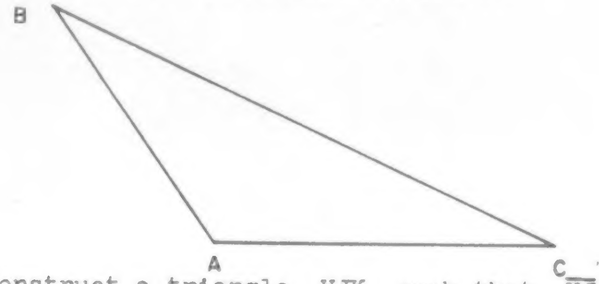
You will recall that this is the construction of Problem 2, Exercises 4-2b.
Is your triangle the same size and shape as triangle ABC?
Use protractor or compass and ruler to check.

Two triangles are congruent if the three sides of one triangle are congruent respectively to the three sides of the other triangle. We will refer to this as Property S.S.S. (Side, Side, Side)

Notice that the congruence sets up a one-to-one correspondence between pairs of sides of two congruent triangles, because we can let congruent sides correspond to each other. That is, suppose we call a, b, and c the sides of one triangle and r, s, and t the sides of a triangle congruent to it and suppose sides a and r, b and s, c and t are congruent. Then we may call a and r corresponding sides, b and s corresponding sides, and c and t corresponding sides. For this one-to-one correspondence it is true that *if two triangles are congruent then their corresponding sides are congruent*. We could set up the same kind of correspondence for angles and have: *If two triangles are congruent then their corresponding angles are congruent*. The converse of the first of these two statements is a true statement, but the converse of the second is not a true statement. You will recall that the converse of the first statement is, if the corresponding sides of two triangles are congruent, the two triangles are congruent. (Property S.S.S.) See Problem 2 below.

Exercises 4-4

Use triangle ABC in Problems 1 and 2.



1. (a) Construct a triangle HJK such that $\overline{HJ} \cong \overline{AB}$, $\overline{JK} \cong \overline{BC}$, and $\overline{HK} \cong \overline{AC}$.
   (b) Construct a triangle HJK such that $\angle H \cong \angle A$, $\angle K \cong \angle C$, and $\overline{HK} \cong \overline{AC}$.
   (c) Construct a triangle HJK such that $\overline{HK} \cong \overline{AC}$, $\overline{HJ} \cong \overline{AB}$, and $\angle H \cong \angle A$.
   (d) Were the triangles you constructed in (a), (b) and (c) congruent to Δ ABC? Why?

2. Construct a triangle HJK, which has $\angle J \cong \angle B$, $\angle H \cong \angle A$, and $\angle K \cong \angle C$, but in which no side is congruent to a side of triangle ABC.

Hint: Choose a segment $\overline{JH}$ which is not congruent to any of the sides of triangle ABC. Construct $\angle J$ and $\angle H$.

(a) Why do we not need to construct $\angle K$?

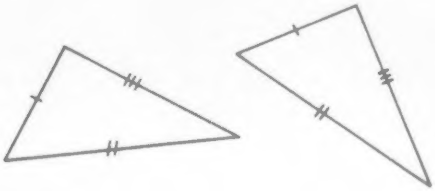
(b) Are triangles ABC and HJK congruent?

(c) Why do we not state a congruent triangles property like those in this section, which we could refer to as Property A.A.A. (Angle, Angle, Angle)?

In **Problems** 3 through 8, there are pairs of triangles. Congruent sides are indicated by the number of strokes on corresponding sides of two triangles, and congruent angles by the number of arcs in corresponding angles. The 2 triangles in a pair may appear to be congruent when they are not.

Use properties S.S.S., S.A.S., or A.S.A. to decide which pairs are congruent and which are not. Identify the property you use in showing the triangles to be congruent if they are.

3.



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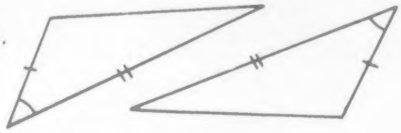

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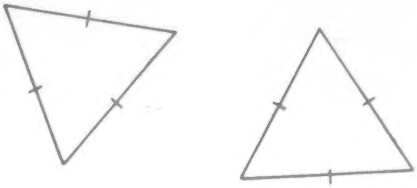

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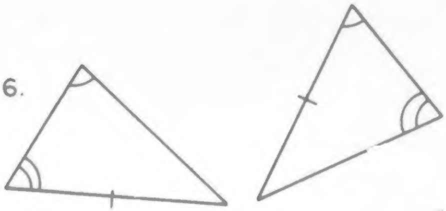

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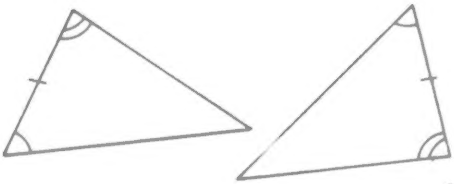

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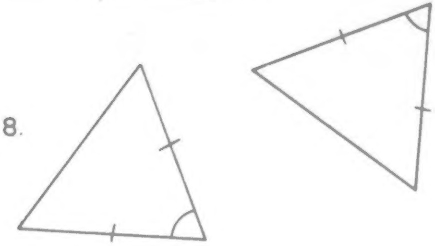


Problems 9 through 17 refer to parts of triangles ABC and PQR. Use Properties S.S.S., S.A.S., or A.S.A. to decide which pairs are congruent and which are not.

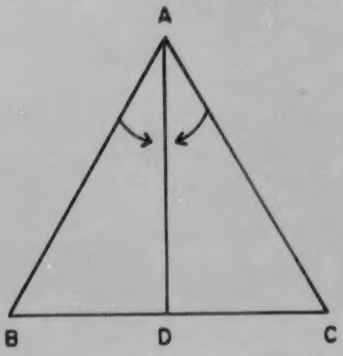
9. $\angle A \cong \angle P$, $\angle B \cong \angle Q$, $\overline{AB} \cong \overline{PQ}$
10. $\overline{AB} \cong \overline{PQ}$, $\overline{AC} \cong \overline{PR}$, $\overline{BC} \cong \overline{QR}$
11. $\angle A \cong \angle B \cong \angle C \cong \angle P \cong \angle Q \cong \angle R$
12. $\angle C \cong \angle R$, $\angle B \cong \angle Q$, $\overline{AB} \cong \overline{PQ}$
13. $\angle A \cong \angle P$, $\angle B \cong \angle Q$, $\overline{BC} \cong \overline{QR}$
14. $AB = 7$, $PQ = 7$, $m(\angle A) = 28$, $m(\angle P) = 28$, $CA = 10$, $RP = 10$
15. $AB = 3$, $BC = 4$, $CA = 5$, $QR = 4$, $PQ = 3$, $RP = 5$
16. $\angle A \cong \angle P$, $\angle B \cong \angle Q$, $\overline{AB} \cong \overline{QR}$
17. $\overline{AB} \cong \overline{AC} \cong \overline{BC}$, $\overline{PQ} \cong \overline{QR}$, $\angle P$ is not congruent to $\angle Q$.
18. If two triangles in the same plane are drawn on opposite sides of a line and are symmetrical with respect to the line, how can one triangle be superimposed on the other?
19. BRAINBUSTER. A triangle has 3 angles and 3 sides. We have seen that two triangles are congruent by Property S.S.S., Property S.A.S., or Property A.S.A. In Problem 2 we saw that two triangles need not necessarily be congruent if the 3 pairs of angles (one from each triangle in a pair) are congruent. Therefore we can not say that if 3 of the 6 parts (angles and sides) of one triangle are congruent to 3 of the 6 parts of another triangle, the triangles are congruent. In Problems 1 and 2 we have considered four cases. There are two cases remaining. What are they? Construct several triangles to show whether in these cases, two triangles are necessarily congruent.

4-5. Showing Two Triangles To Be Congruent

If we wish to know whether two triangles are congruent, we could measure each side and each angle. This would require 12 measurements, 6 for each triangle. One way we could cut down on the amount of work would be to measure only two angles in each triangle. Since the sum of the measures of the angles of a triangle is 180 we could determine the third angle in each triangle without measurement. This would leave only 10 measurements if we were trying to see if two triangles are congruent. Of course we could also test for congruence by "cutting and fitting," but this is also time-consuming and often inconvenient.

We can shorten our use of measurement if we make use of one of the three properties of congruent triangles. For example if we find two pairs of sides (one from each triangle in a pair) in two triangles are congruent, we would then measure the included angle in each triangle. If the included angles are found to be congruent, then we need take no further measurements. Property S.A.S. tells us that the two triangles are congruent.

In the figure, $\overline{AB} \cong \overline{AC}$. Hence, the triangle is isosceles. $\angle DAB \cong \angle DAC$. Hence, $\overline{AD}$ bisects $\angle BAC$. Is D the midpoint of $\overline{BC}$? It looks as if it might be. We could make measurements which would answer this question if we allow for the approximate nature of measurement. Would the bisector of the angle determined by the congruent sides of any isosceles triangle always intersect the opposite side in the midpoint of that side?



We can find an answer to the question if we make use of one of our congruent triangles properties. Consider Δ DAB and Δ DAC.

$\overline{AB} \cong \overline{AC}$ by construction

$\angle DAB \cong \angle DAC$ AD is on the bisector of $\angle BAC$

$\overline{AD} \cong \overline{AD}$ AD is in both triangles

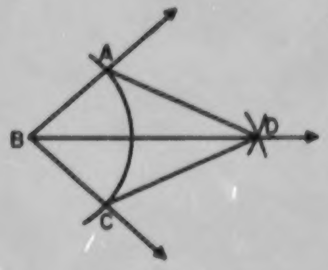
By Property S.A.S., $\triangle DAB \cong \triangle DAC$, since 2 sides and the included angle of $\triangle DAB$ are congruent to the corresponding sides and corresponding angle of $\triangle DAC$.

$\overline{BD} \cong \overline{DC}$ since they are corresponding sides of congruent triangles. Therefore, D is the midpoint of $\overline{BC}$. If Property S.A.S. is true, we now can be sure that:

The bisector of the angle determined by the congruent sides of an isosceles triangle intersects the third side in the midpoint of that side.

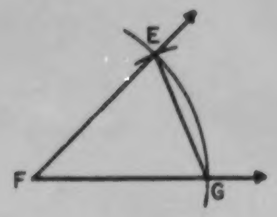
Exercises 4-5a

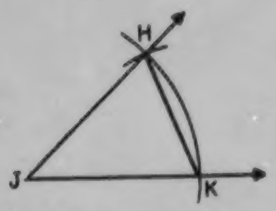
1. In the figure the construction of the bisector of $\angle ABC$ is shown. Two segments, $\overline{AD}$ and $\overline{CD}$ are drawn.



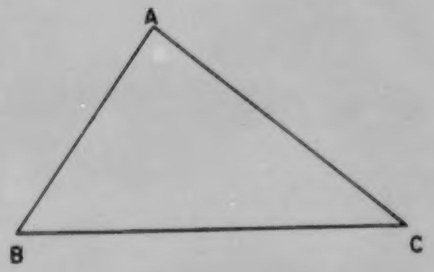
   (a) What parts of triangle ABD are congruent to corresponding parts of triangle CBD by construction?
   (b) Is triangle ABD congruent to triangle CBD? Why?
   (c) Is $\angle ABD$ congruent to $\angle CBD$? Why?

2. In the figures shown here the construction of $\angle HJK$ makes $\angle HJK \cong \angle EFG$. Segments $\overline{EG}$ and $\overline{HK}$ are drawn.

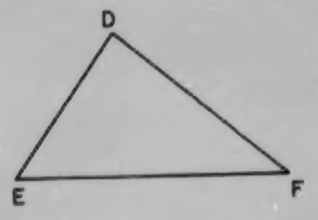




   (a) What parts of triangle EFG are congruent to corresponding parts of triangle HJK by construction?
   (b) Is triangle EFG congruent to triangle HJK? Why?
   (c) Is $\angle J$ congruent to $\angle F$? Why?



3. Use your protractor to find the measures in degrees of the 3 angles in each triangle.
   (a) Are there some pairs of congruent angles? If so, list them.
   (b) Could we say that the triangles are congruent? Why?
   (c) Suppose that triangle DEF were constructed with $\angle A \cong \angle D$ and $\angle B \cong \angle E$, and with $\overline{ED}$ the same length as $\overline{AB}$. What would be true about the two triangles? Why?

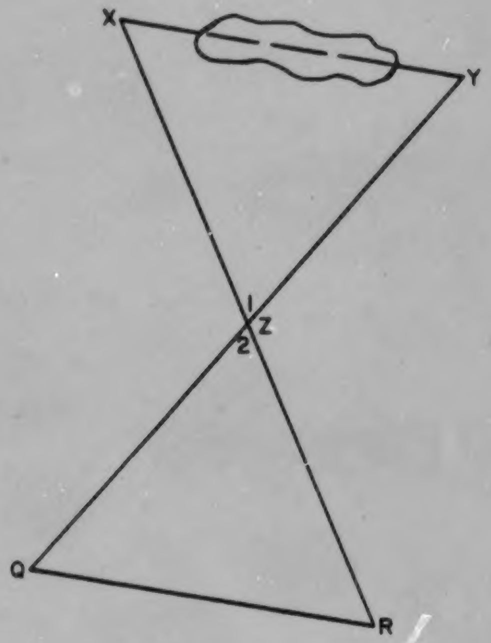


4. Mr. Thompson wishes to measure the distance between two posts on edges of his property. A grove of trees between the two posts (X and Y) makes it impossible to measure the distance XY directly. He locates point Z such that he can lay out a line from X to Z and continue it as far as needed.

Point Z is also in a position such that Mr. Thompson can lay out a line $\overleftrightarrow{YZ}$ and continue it as far as needed. Mr. Thompson knows that $\angle 1 \cong \angle 2$ since they are vertical angles. He extends $\overline{YZ}$ so that $\overline{QZ} \cong \overline{YZ}$, and $\overline{XZ}$ so that $\overline{XZ} \cong \overline{RZ}$.

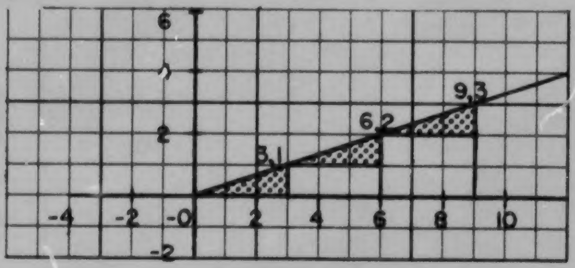
(a) Are triangles XYZ and QZR congruent? Why?

(b) How can Mr. Thompson determine the length of XY?



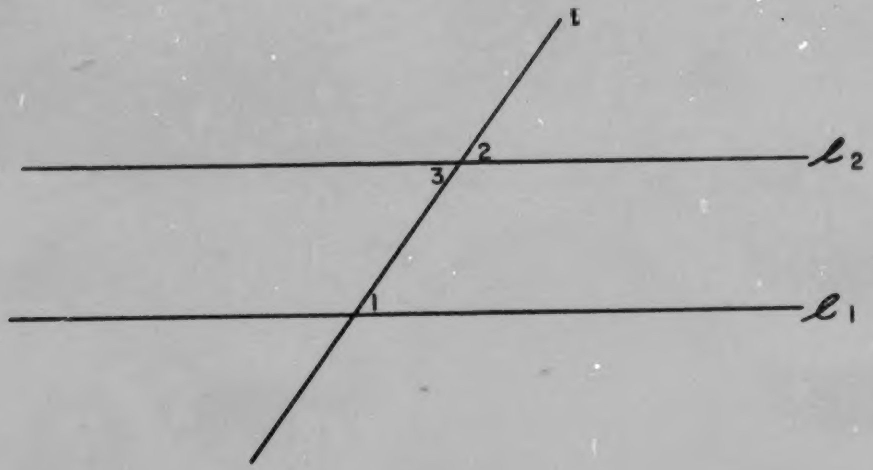
5. (a) In the figure below are the shaded triangles congruent?

(b) Which properties do you use to test the congruency of these triangles?



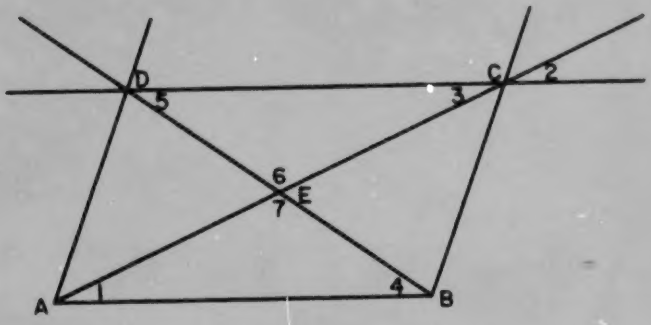
We will make use of this figure again when we study similar triangles.

6. Line ℓ_1 and line ℓ_2 are parallel lines cut by a transversal t.



(a) What do you know about angles 1 and 2?
(b) Are angles 2 and 3 congruent? Why?
(c) Show that $\angle 1 \cong \angle 3$.

7. In the parallelogram, ABCD, the diagonals $\overline{AC}$ and $\overline{BD}$ intersect at E.



(a) Is angle 1 (in Δ ABE) congruent to angle 2?

(b) What kind of angles are $\angle 2$ and $\angle 3$? Are they congruent?

(c) How does the size of $\angle 1$ compare with the size of $\angle 3$?

(d) Show that $\angle 6 \cong \angle 7$ and $\angle 5 \cong \angle 4$.

(e) When two triangles have three pairs of congruent angles, are the triangles always congruent? If not, what else is needed?

(f) Is any side of Δ ABE congruent to the corresponding side of Δ CDE?

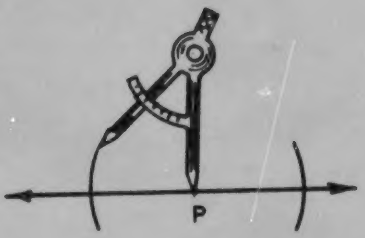
(g) Show that the diagonals of a parallelogram bisect each other.

*8. If two angles of a triangle are congruent, the sides opposite them are congruent. Use a property of congruent triangles to show that this statement is true.

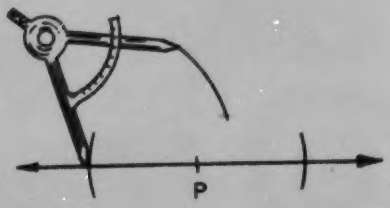
Hint: A triangle can be congruent to itself.
Show ABC $\cong$ BAC, where $\angle A \cong \angle B$.

Erecting a perpendicular from a point on a line

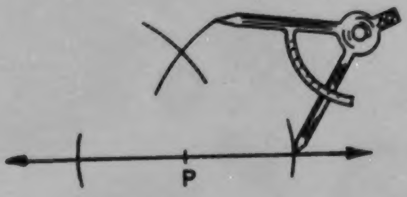
Follow the four steps in the construction as illustrated in the figures, and construct a perpendicular from a point on a line. Measure the angles in the figure you construct to see if the line you have constructed appears to be a perpendicular.



If the segments $\overline{GJ}$ and $\overline{HJ}$ are drawn, two triangles GPJ and HPJ are formed.

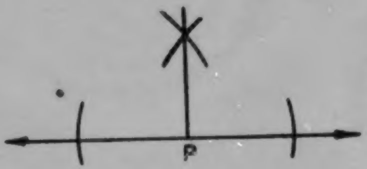


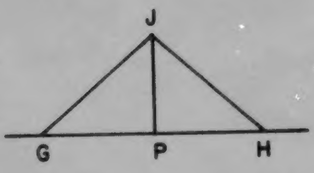
$\overline{GP} \cong \overline{HP}$	by construction.
$\overline{GJ} \cong \overline{HJ}$	by construction.
$\overline{PJ} \cong \overline{PJ}$	a common side to both triangles.
$\triangle GPJ \cong \triangle HPJ$	Property S.S.S.
$\angle GPJ \cong \angle HPJ$	corresponding angles of two congruent triangles.



If a protractor is laid along $\overrightarrow{PH}$ with the vertex mark at P and 0° mark on $\overrightarrow{PH}$, then the 180° mark will be on the ray $\overrightarrow{PG}$. This means that the sum of the measures of the angles at P is 180 and, since these measures are equal, each must be 90. Hence, angles JPG and JPH are right angles.

You have now seen how to construct the perpendicular to a line at a point on the line, and you have seen why the construction "works."





Exercises 4-5b

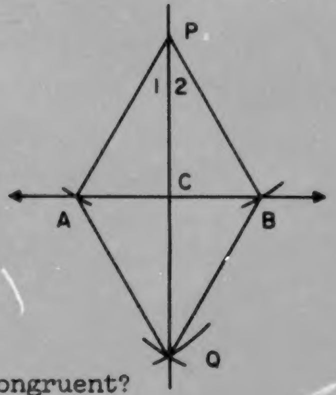
1. Draw a segment about 4 inches in length, and then construct perpendiculars to the segment at each endpoint.

   Hint: Extend the line segment when necessary.

2. Construct a perpendicular to a line from a point not on the line.

   Hint: Follow the steps in the constructions given on the preceding page.

3. Your construction in Problem 2 will look like the figure at the right. Label the intersection of the 2 arcs Q and the intersection of $\overleftrightarrow{PQ}$ and $\overleftrightarrow{AB}$ as C.



   (a) Why is triangle APQ congruent to triangle BPQ?

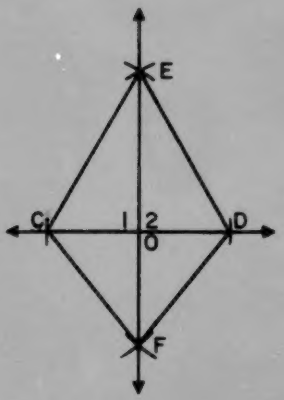
   (b) Why are angles 1 and 2 congruent?

   (c) What other pairs of angles are congruent?

4. Using congruent triangles in the figure in Problem 3, show that $\overleftrightarrow{PQ}$ is perpendicular to $\overleftrightarrow{AB}$.

   Hint: Use Property S.A.S. to show triangle ACP is congruent to triangle BCP. Use a protractor to find the sum of the measures of $\angle ACP$ and $\angle BCP$.

5. The construction of the perpendicular bisector of segment $\overline{CD}$ is shown. Usually the same radius is used for the four arcs. However, it is only necessary for the two arcs that intersect on one side of the segment to have equal radii. Thus, the arcs drawn from C and D that intersect at E have equal radii, and the two arcs



drawn from C and D that intersect at F have equal radii. In the figure, CE is not congruent to $\overline{CF}$. By applying some of the properties about congruent triangles, show why $\overline{EF}$ bisects $\overline{CD}$ and is perpendicular to $\overline{CD}$.

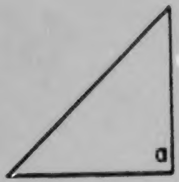
6. (a) How many pairs of congruent triangles are there in the figure for Problem 5? List them by pairs.
   (b) List the pairs of corresponding sides for each pair of congruent triangles.
   (c) List the pairs of corresponding angles for each pair of congruent triangles.

7. (a) Draw a triangle. Then erect perpendiculars from each vertex to the opposite side. Extend the perpendiculars until they intersect. (It may be necessary to extend a side of the triangle so that the perpendicular will meet this side.)
   (b) What do you notice in this figure?

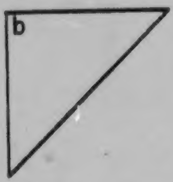
8. In what ways are the constructions for bisecting a line segment (Section 4-2) and constructing the perpendicular to a line segment alike, and in what way are they different?

4-6. The Right Triangle

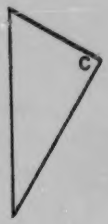
Triangles may be named according to the measures of the angles. In the following set of triangles each has an angle whose measure in degrees is 90. Triangles having this property are called right triangles.



m ($\angle$ a) is 90.



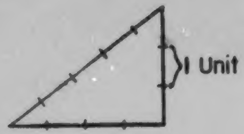
m ($\angle$ b) is 90.



m ($\angle$ c) is 90.

The ancient Egyptians are said to have used a particular right triangle to make corners "square." This triangle has sides 3 units long, 4 units long, and 5 units long. When such a triangle is made of tautly stretched rope, the angle between the two shorter sides is a right angle.

While the Egyptians are thought to have made use of this fact, it was left to the Greeks to prove the relationship involved. The Greek philosopher and mathematician, Pythagoras, who lived about 500 B.C. became interested in the problem. Pythagoras is credited with the proof of the basic property which we will study in this section; this property is still known by his name, the Pythagorean Property.



It is thought that Pythagoras looked at a mosaic like the one pictured in Figure 4-6A. He noticed that there are many triangles of different sizes that can be found in the mosaic. But he noticed more than this. If each side of any triangle is used as one side of a square, the sum of the areas of the two smaller squares is the same as the area of the larger square. In Figure 4-6B, two triangles of different size are inked in and the squares drawn on the sides of the triangles shaded.

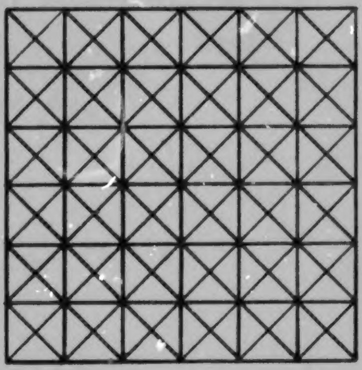


Figure 4-6A

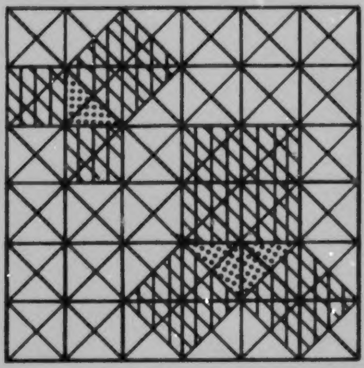
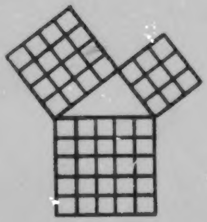


Figure 4-6B

Count the number of the smallest triangles in each square. For each triangle that is inked in, how does the number of small triangles in the two smaller squares compare with the number in the larger square? If you draw a mosaic like this, you will find that this is true not only for the two triangles given here but for even larger triangles in this mosaic.

Pythagoras probably noticed the same relation in the 3-4-5 triangle that the Egyptians had used for so long to make a right angle. The small squares are each 1 square unit in size. In the three squares there are 9, 16 and 25 small squares. Notice that 9 + 16 = 25. Pythagoras was able to prove that in any right triangle, the area of the square on the hypotenuse (longest side) is equal to the sum of the areas of the squares on the other two sides. This is the Pythagorean Theorem or, as we shall call it, the Pythagorean Property.



So far this has been shown only for two very special right triangles. It is true for all right triangles. Some of you may try to prove it for yourselves by studying Section 4-7.

Exercises 4-6a

1. Using your straightedge and protractor draw the following:
   (a) 30-60 triangle
   (b) 45-45 triangle
   (c) 70-20 triangle

2. Show for each set that the square of the first number is equal to the sum of the squares of the other two numbers.

   (a) 5, 4, 3 (c) 25, 7, 24

   (b) 13, 5, 12 (d) 20, 16, 12

3. Make a drawing of the triangles with the sides of length given in part (a) of Problem 2. Use your protractor to show that this triangle is a right triangle.

4. Draw right triangles, the lengths of whose shorter sides (in centimeters) are:

   (a) 1 and 2 (b) 4 and 5 (c) 2 and 3.

   Measure, to the nearest one-tenth of a centimeter if possible, the lengths of the hypotenuses of these triangles.

5. Use the Pythagorean Property to find the area of the square on the hypotenuse for each triangle in Problem 4.

In the last set of exercises, you worked with sides of right triangles and squares built on the sides of right triangles. In Problem 4 you measured the length of the hypotenuse. The hypotenuse is the side opposite the right angle. It is not very useful to know the area of these squares, but many uses can be made of the Pythagorean Property if we can use it to find the length of the third side when we know the length of two sides. In mathematical language the Pythagorean Property is $c^2 = a^2 + b^2$ where c stands for the measure of the hypotenuse and a and b stand for the measures of the other two sides. The measures of any two sides can be substituted in the number sentence above, and from this the third value can be found. We can use a familiar triangle to show this. If the two short sides are 3 units and 4 units, what is the square of the hypotenuse?

$$c^2 = a^2 + b^2$$

$$c^2 = 3^2 + 4^2$$

$$c^2 = 9 + 16$$

$$c^2 = 25.$$

Since c^2 is equal to 25, we will know c if we can find a number whose product is 25 when it is multiplied by itself. Of course, $5 \times 5 = 25$, so $c = 5$; 5 is the positive square root of 25. If a number is the product of two equal factors, then each factor is a square root of the number. The symbol for the positive square root is: $\sqrt{}$. The numeral is placed under the sign; for example, $\sqrt{25} = 5$.

What is $\sqrt{9}$? $\sqrt{16}$? $\sqrt{36}$? $\sqrt{30}$? The first three are easy to understand since $3 \times 3 = 9$, $4 \times 4 = 16$, and $6 \times 6 = 36$, but there is no integer that can be multiplied by itself to give the product 30. In fact, there is no rational number whose square is 30!

Can you find a number multiplied by itself that will give a product close to 30? Yes, $5 \times 5 = 25$ which is close to 30. What is 6×6 or 6^2? This tells us that $\sqrt{30}$ is greater than 5 but less than 6. We might try to get a closer approximation by squaring 5.1, 5.2, 5.3, 5.4 and 5.5. Now $(5.4)^2 = 29.16$ and $(5.5)^2$ is equal to 30.25. Because 30.25 is closer to 30 than 29.16 we might assume that $\sqrt{30}$ is nearer 5.5 than 5.4. However we would need to square 5.45 to get 29.7025 before we can say that $\sqrt{30}$ is 5.5 to the nearest tenth.

You may want to estimate the square roots of some numbers this way or use the table at the end of this section.

Note that in the table at the end of this section, $\sqrt{30}$ is 5.477 to the nearest thousandth. Rounded to tenths this would be 5.5. You can see how close your estimate is to the number in the table.

The table gives the decimal value (to the nearest thousandth) that is closest to the square root of integers from 1 to 100. You can also use the table to find the square root of all counting numbers up to 10,000 that have rational square roots.

Exercises 4-6b

When approximate values are used in these problems, use the symbol, $\approx$, in the work and answer.

1. Use the table to find the approximate value of:

   (a) $\sqrt{5}$ (c) $\sqrt{13}$ (e) $\sqrt{676}$

   (b) $\sqrt{41}$ (d) $\sqrt{92}$ (f) $\sqrt{5625}$

2. Use the Pythagorean Property to find the length of the hypotenuse for each of these triangles.

   (a) Length of a is 1", length of b is 2"
   (b) Length of a is 4', length of b is 5'
   (c) Length of a is 2", length of b is 3"
   (d) Length of a is 5 yd. and the length of b is 6 yd.
   (e) Length of a is 3 ft. and the length of b is 9 ft.
   (f) Length of a is 1 unit and the length of b is 3 units.

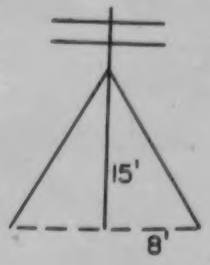
3. Sometimes the hypotenuse and one of the shorter sides is known. How can you find the length of the other side? As an example, use this problem. The hypotenuse of a right triangle is 13 ft. and one side is 5 ft. Find the length of the third side.

$$\begin{aligned} c^2 &= a^2 + b^2 \\ 13^2 &= 5^2 + b^2 \\ 13^2 + {}^{-}(5^2) &= b^2 \qquad \text{(addition property of equality)} \\ 169 + {}^{-}(25) &= b^2 \\ 144 &= b^2 \\ \sqrt{144} &= \sqrt{b^2} \\ 12 &= b \end{aligned}$$

   The third side is 12 feet long. Find the third side of these right triangles. The measurements are in feet.

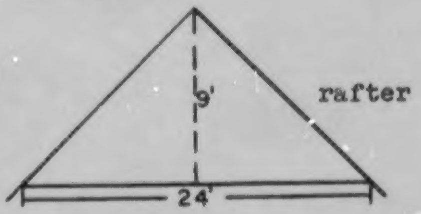
   (a) $c = 15,\ b = 9$
   (b) $c = 26,\ a = 24$
   (c) $c = 39,\ b = 15$

4. A telephone pole is steadied by guy wires as shown. Each wire is to be fastened 15 ft. above the

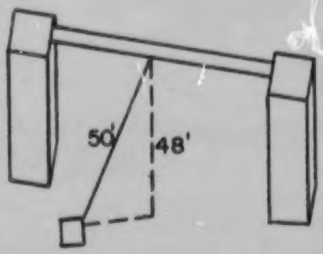


ground and anchored 8 ft. from the base of the pole. How much wire is needed to stretch one wire from the ground to the point on the pole at which the wire is fastened?

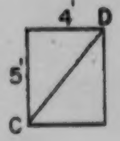
5. A roof on a house is built as shown. How long should each rafter be if it extends 18 inches over the wall of the house?



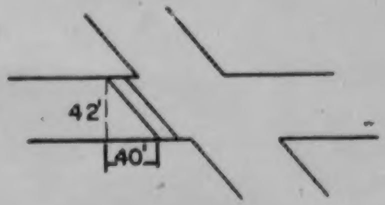
6. A hotel builds an addition across the street from the original building. A passageway is built between the two parts at the third-floor level. The beams that support this passage are 48 ft. above the street. A crane operator is lifting these beams into place with a crane arm that is 50 ft. long. How far down the street from a point directly under the beam should the crane cab be?



7. A garden gate is 4 ft. wide and 5 ft. high. How long should the brace that extends from C to D be?



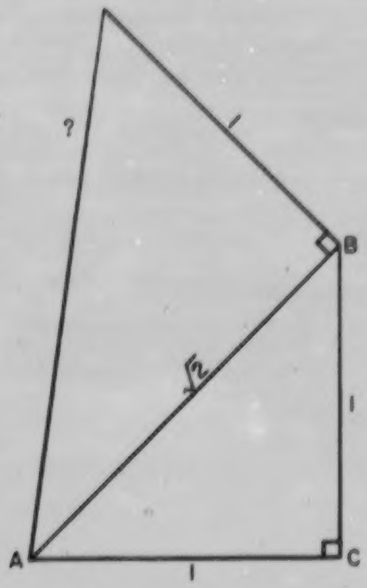
8. Two streets meet at the angle shown. The streets are 42 ft. wide. Lines for a cross-walk are painted so that the walk runs in the same direction as the street. If it is 40 ft. from one end of the cross-walk to the point that is on the perpendicular from the other end point of the walk, how long is the cross-walk?



9. How long is the throw from home plate to 2nd base in a softball game? The bases are 60 ft. apart, and a softball diamond is square in shape. Give your answer to the nearest whole foot.

*10. Draw a square whose sides are of length 1 unit. What is the length of the diagonal? Check by measurement. Now draw a right triangle with the sides 1 unit long. What is the length of the hypotenuse?

*11. Now draw a right triangle of sides "square root of 2" and 1 units in length as shown in the figure. In the figure the measure of the length of $\overline{AB}$ is the square root of 2. What is the length of the hypotenuse of this new triangle?



TABLE

SQUARES AND SQUARE ROOTS OF NUMBERS

Number	Squares	Square roots	Number	Squares	Square roots
1	1	1.000	36	1,296	6.000
2	4	1.414	37	1,369	6.083
3	9	1.732	38	1,444	6.164
4	16	2.000	39	1,521	6.245
5	25	2.236	40	1,600	6.325
6	36	2.449	41	1,681	6.403
7	49	2.646	42	1,764	6.481
8	64	2.828	43	1,849	6.557
9	81	3.000	44	1,936	6.633
10	100	3.162	45	2,025	6.708
11	121	3.317	46	2,116	6.782
12	144	3.464	47	2,209	6.856
13	169	3.606	48	2,304	6.928
14	196	3.742	49	2,401	7.000
15	225	3.873	50	2,500	7.071
16	256	4.000	51	2,601	7.141
17	289	4.123	52	2,704	7.211
18	324	4.243	53	2,809	7.280
19	361	4.359	54	2,916	7.348
20	400	4.472	55	3,025	7.416
21	441	4.583	56	3,136	7.483
22	484	4.690	57	3,249	7.550
23	529	4.796	58	3,364	7.616
24	576	4.899	59	3,481	7.681
25	625	5.000	60	3,600	7.746
26	676	5.099	61	3,721	7.810
27	729	5.196	62	3,844	7.874
28	784	5.292	63	3,969	7.937
29	841	5.385	64	4,096	8.000
30	900	5.477	65	4,225	8.062
31	961	5.568	66	4,356	8.124
32	1,024	5.657	67	4,489	8.185
33	1,089	5.745	68	4,624	8.246
34	1,156	5.831	69	4,761	8.307
35	1,225	5.916	70	4,900	8.367

Number	Squares	Square roots
71	5,041	8.426
72	5,184	8.485
73	5,329	8.544
74	5,476	8.602
75	5,625	8.660
76	5,776	8.718
77	5,929	8.775
78	6,084	8.832
79	6,241	8.888
80	6,400	8.944
81	6,561	9.000
82	6,724	9.055
83	6,889	9.110
84	7,056	9.165
85	7,225	9.220
86	7,396	9.274
87	7,569	9.327
88	7,744	9.381
89	7,921	9.434
90	8,100	9.487
91	8,281	9.539
92	8,464	9.592
93	8,649	9.644
94	8,836	9.695
95	9,025	9.747
96	9,216	9.798
97	9,409	9.849
98	9,604	9.899
99	9,801	9.950
100	10,000	10.000

4-7. *One Proof of the Pythagorean Property

There are many proofs of this property. The one used here is not the one used by Pythagoras. You should actually draw and cut the squares called for in the explanation.

Draw two squares the same size. Separate the first square into two squares and two rectangles as shown here:

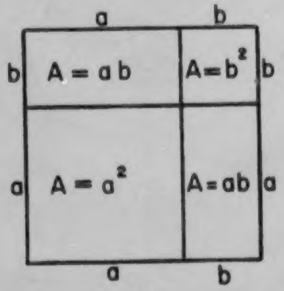


Figure 4-7A

Let the measure of each side of the larger square in Figure 4-7A be a and the measure of each side of the small square be b. Notice the areas of the small squares and rectangles.

One square has an area of measure a^2.
The other square has an area of measure b^2.
Each rectangle has an area of measure ab.

Since the area of Figure 4-7A is equal to the sum of the areas of all of its parts, the measure of the area of Figure 4-7A is

$$a^2 + 2(ab) + b^2.$$

Class Exercises 4-7

1. Let $a = 4$ and $b = 3$. Show that $(a + b)^2 = a^2 + 2ab + b^2$ for these numbers.

2. Let $a = 2$ and $b = 6$ and check the same relationship.

Now turn to the second square. Use the same numbers, a and b, that were used in the first square.

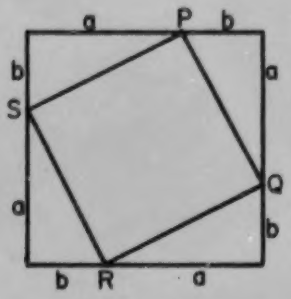


Figure 4-7B

Mark the lengths off as shown here and draw the segments $\overline{PQ}$, $\overline{QR}$, $\overline{RS}$ and $\overline{SP}$. The large square is separated into 4 triangles and a quadrilateral that appears to be a square.

The measure of each triangular area is $\frac{1}{2}$ ab. There are four congruent triangles. The sum of the measures of the areas of all four triangles is $4(\frac{1}{2}ab)$ or 2 ab

If you look back to Figure 4-7A you will see that 2ab is the measure of the area of the two rectangles. Cut the two rectangles from the first square. Cut along the diagonal of each rectangle. See if the four triangles you cut are congruent with those in the second square. See if you can follow these steps.

$$A_{square} = a^2 + 2ab + b^2 \quad \text{(from Figure 4-7A)}$$

$$A_{square} = 4(\tfrac{1}{2}ab) + A_{PQRS} \quad \text{(from Figure 4-7B)}$$

$$= 2ab + A_{PQRS}$$

Therefore $a^2 + 2ab + b^2 = 2ab + A_{PQRS}$ Why?

$a^2 + b^2 = A_{PQRS}$ (addition property of equality.)

This shows that PQRS has an area whose measure is $a^2 + b^2$ units, but a^2 is the measure of the area of one small square in the first figure and b^2 is the measure of the area of the other square. From this the area of the figure in the center of the second square is equal to the sum of the areas of the two small squares.

Place the square whose area measure is a^2 along the side of length a of one triangle in the second square. Place the square whose area measure is b^2 along the side of length b of the same triangle. The areas of the squares on the two sides of the triangle are equal to the area of the figure in the center of Figure 4-7B. All we need to do now is to prove that this figure is a square!

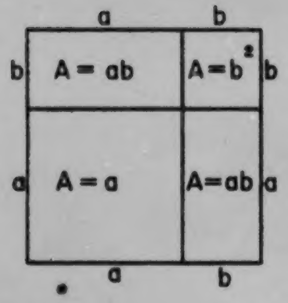
What are the properties of a square?

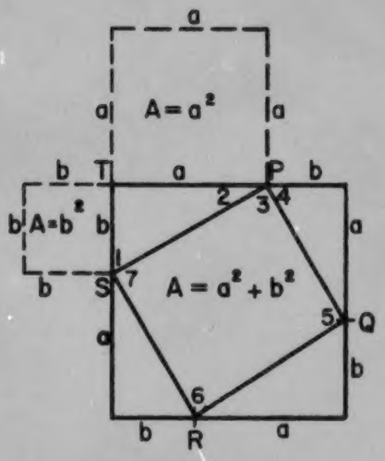
1. The four sides are congruent.
2. Each angle is 90° in measurement.

If we can prove these two conditions for the quadrilateral in Figure 4-7B, the Pythagorean Property has been proved.

As has been stated, the four triangles are congruent since for each pair two corresponding sides and the angles determined by these sides are congruent. As a result, PQ = QR = RS = SQ because they are measures of corresponding segments of congruent triangles.

So far we have shown that the squares in Figure 4-7A are congruent to the squares on the short sides of any one of the triangles in Figure 4-7B. We have also shown that the sum of the areas of these squares is equal to the area of PQRS, and that PQRS has four congruent sides. Let us prove that the angles are right angles.


a
b
b
A = ab
A=b²
b
a
A = a
A=ab
a
a
b


a
a
A = a²
a
b
T
a
P
b
2
3
4
b
A=b²
b
a
1
b
S
7
A = a² + b²
5
Q
a
b
6
b
R
a

(1) In Δ PST, m(∠ 1) + m(∠ 2) = 90 Why?

(2) m(∠ 1) = m(∠ 4) Why?

(3) Therefore m(∠ 4) + m(∠ 2) = 90 Why?

(4) and m(∠ 2) + m (∠ 3) + m(∠ 4) = 180 Why?

(5) m(∠ 3) + 90 = 180 Why?

(6) and m(∠ 3) = 90 Why?

We can go through the same type of reasoning to show that angles 5, 6 and 7 are also right angles.

PQRS has been proved to be a square and its area has also been proved equal to the sum of the areas of the squares on the other two sides.

4-8. Quadrilaterals

Symmetry and congruence can be found in some quadrilaterals or parts of quadrilaterals. It is also possible to find applications of the Pythagorean Property in quadrilaterals. This section has problems based on quadrilaterals that make use of these three ideas. In the Exercises it will be of assistance to keep in mind that:

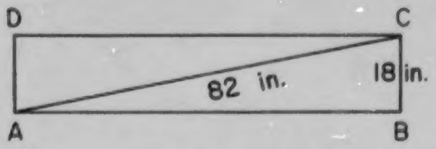
> A trapezoid has only one pair of parallel sides.
> A parallelogram has two pairs of parallel sides.
> A parallelogram which has four right angles is a rectangle.
> A rectangle with four congruent sides is a square.

Exercises 4-8

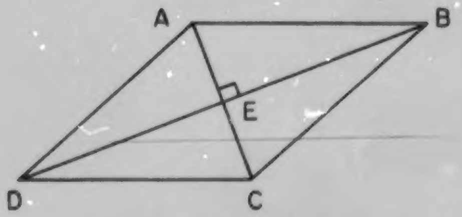
1. A figure is symmetrical with respect to a line if it has that line as an axis of symmetry. Which of these figures are always symmetrical with respect to a line?
   (a) Trapezoid
   (b) Parallelogram
   (c) Rectangle
   (d) Square
2. How many axes of symmetry are there in:
   (a) A rectangle?
   (b) A square?
3. Is is possible to draw a trapezoid that is symmetrical with respect to a line? If so, draw one.
4. Is it possible to draw a parallelogram that is symmetrical with respect to a line? If so, draw one.
5. One diagonal of a quadrilateral separates the figure into two triangles.
   (a) What is the sum of the measures of the angles of a quadrilateral?

(b) Name the quadrilaterals that are separated into congruent triangles by a diagonal.

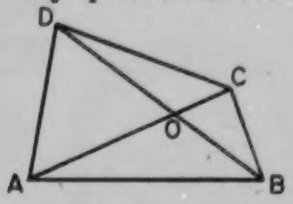
6. Rectangle ABCD has a diagonal that is 82 inches in length. The width is 18 inches. What is the length?



7. The diagonals of this parallelogram are perpendicular and the sides are equal in measure. The shorter diagonal is 14 feet in length and the longer one is 48 feet. How long is each side? (Hint: The diagonals of a parallelogram bisect each other.)

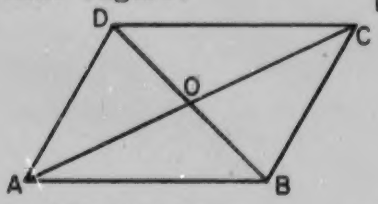


8. Two diagonals have been drawn in each of the following figures. For each figure, A, B, C, D, E, and F, answer all questions. Answer questions (a) and (b) for one figure before you answer any questions for the next figure.



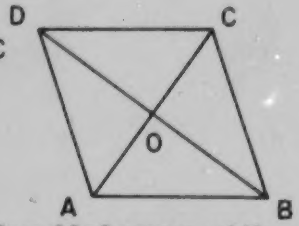
Quadrilateral

Figure A



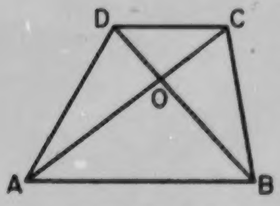
Parallelogram

Figure B



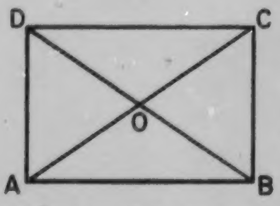
Parallelogram with all sides congruent

Figure C



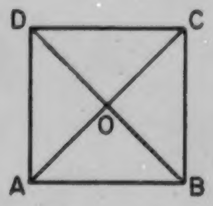
Trapezoid

Figure D



Rectangle

Figure E



Square

Figure F

(a) How many triangles are there in the figure?

(b) Do any pairs of triangles appear to be congruent? If so, name the triangles in pairs.

(c) If you found triangles ABC and triangles CDA congruent, in one or more figures, choose one figure and show why they are congruent. Use the properties, S.S.S., S.A.S., or A.S.A. to show this. If you found triangles ABO and CDO congruent do the same for these triangles. If you found both sets congruent in some figure, choose which pair you wish to show congruent.

4-9. Solids

You have been drawing figures which are contained within a plane. You are now to practice drawing on the surface of your paper pictures of figures in space. You have found that it is easy to draw a plane figure on the surface of your paper or on the chalkboard. You will find that it is not so easy to draw pictures of solids on paper or on the chalkboard. This is because you must draw the figure on a surface in such a way that it will appear to have depth. In other words, you want to make a drawing on your paper have the appearance of a box. This requires the use of projection which you have possibly studied in art.

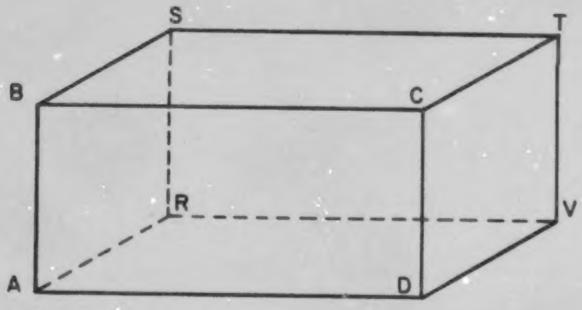
I. Right Prisms

(1) Rectangular Right Prisms. A good example of a rectangular prism is a cereal box. One way to draw a box is as follows:

(a) Draw a rectangle such as ABCD in the figure below.

(b) Now draw a second rectangle RSTV in a position similar to the one in the figure. You may wish to use dotted line segments in parts of this figure.

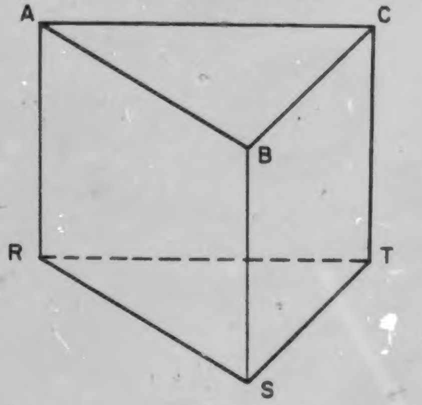
(c) Draw $\overline{AR}$, $\overline{BS}$, $\overline{DV}$, and $\overline{CT}$.



When you look at a solid you cannot see all of the edges, or faces, unless the solid is transparent. For this reason we **represent the edges, which are not visible, by dotted line segments.** This also helps to give the proper perspective to the drawing. If you wish, the dotted line segments do not have to be drawn.

(2) Triangular Right Prisms. Now that you have drawn a rectangular prism, a triangular prism will be easy.

- (a) Draw any triangle ABC.
- (b) At points A and C draw lines of equal measure perpendicular to $\overline{AC}$. Label the end points of these perpendiculars as R and T.
- (c) Draw $\overline{BS}$ parallel to $\overline{AR}$ and of equal measure with $\overline{AR}$.
- (d) Draw $\overline{RS}$ and $\overline{ST}$. Then $\overline{TR}$ may be drawn with a dotted line.
- (e) Compare your figure with the one below.



(f) How many faces does this solid have?

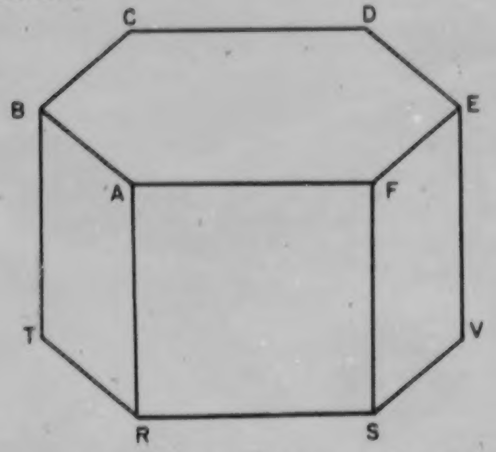
(g) In what way are the faces different?

(3) Hexagonal Right Prisms

(a) Draw a hexagon, similar to ABCDEF of the figure below. This is not intended to be a regular hexagon. In order to get the proper perspective it may be necessary to draw some sides longer than other sides. This is the way a hexagon would appear if looked at from an angle.

(b) Draw at A and F perpendiculars to $\overline{AF}$ having equal measures. Label the end points R and S.

(c) Draw $\overline{BT}$ and $\overline{EV}$ parallel to $\overline{AR}$ and of equal measure with $\overline{AR}$. If you wish, you may draw dotted lines to represent the edges which are not visible.



(d) How many faces does this solid have?

(e) In what way are the faces different?

II. Pyramids

Perhaps this is the first time you have heard of the set of solids, called pyramids, in a mathematics text. You have probably heard of the famous Pyramids of Egypt. A pyramid has one base, which is a region formed by a polygon, and triangular faces which are made by joining the vertices of the polygon to a point which is not in the plane of the polygon. A more accurate description of pyramids will come in a later chapter. Let us draw one.

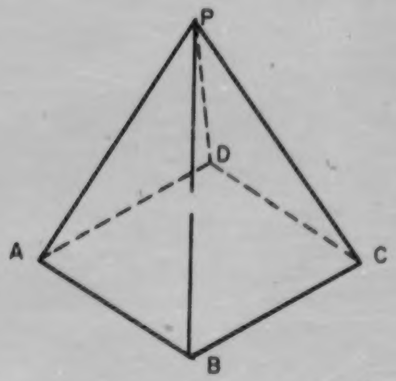
(a) In this drawing let the base represent a square with a vertex at the bottom of the drawing. Here again, you must be careful to get the proper perspective.

(b) First, draw only two sides of the square, such as $\overline{AB}$ and $\overline{BC}$, as shown in the figure below.

(c) Now select a point P, directly above point B, and draw $\overline{PA}$, $\overline{PB}$ and $\overline{PC}$.

(d) $\overline{AD}$, $\overline{CD}$ and $\overline{PD}$ may now be drawn as dotted line segments intersecting at D, with $\overline{AD}$ parallel to $\overline{BC}$ and $\overline{CD}$ parallel to $\overline{AB}$.

(e) How many faces does this pyramid have?



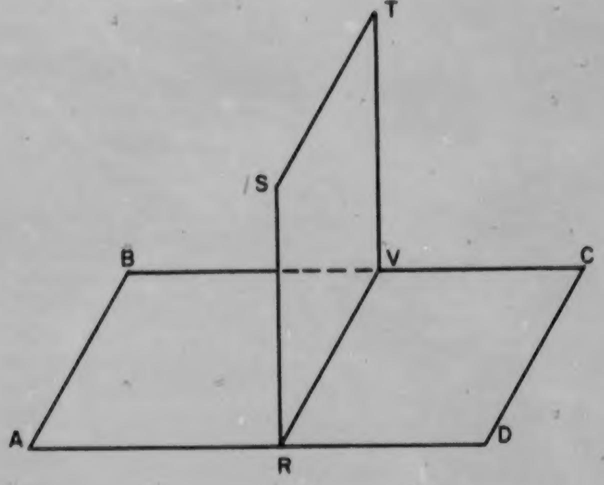
III. Intersecting Planes

There are times when it is useful to represent, on a surface, the intersection of two or more planes. An example is the intersection of a wall and the floor of your room. Another example is seen when you open your book and hold up a page or so. Drawing such a representation is not too difficult, and it becomes easier with practice. Look at the figure below and follow the instructions in drawing a figure of your own.

(a) Draw a parallelogram ABCD.

(b) Select a point R on $\overline{AD}$ and draw $\overline{RV}$ parallel to $\overline{AB}$.

(c) Draw a perpendicular $\overline{VT}$ to $\overline{BC}$ and a perpendicular $\overline{RS}$ to $\overline{AD}$ so that $\overline{VT}$ and $\overline{RS}$ have equal measure. Now draw $\overline{TS}$.



(d) What kind of a figure is RSTV?

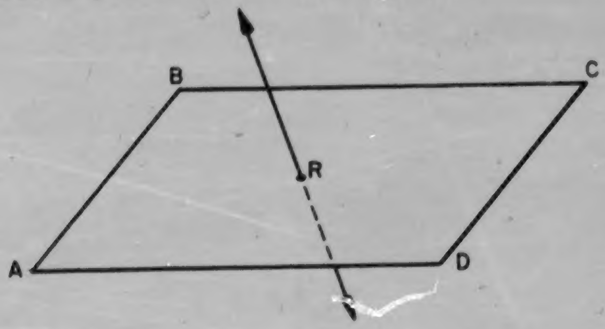
(e) It is not necessary that $\overline{VT}$ and $\overline{RS}$ be perpendicular as described above. It is desirable, however, that RSTV be a parallelogram.

IV. Line Intersecting a Plane

This type of drawing is also useful at times. It is illustrated below.

(a) Draw a parallelogram such as ABCD.

(b) Select a point R on the surface of ABCD.



(c) Now draw a line through R so that it appears to pass through the surface of ABCD. This will require some practice.

(d) You will have a better picture if the line through R is not parallel to a side of the parallelogram.

Exercises 4-9

1. Draw a rectangular prism so that it will appear to be tall and slender.
2. Draw a triangular prism so that the triangular faces will appear to be right triangles.
3. Draw a pentagonal prism.
4. Draw a rectangular prism so that it will appear to be short and fat.
5. Draw a pyramid with the base a quadrilateral which does not appear to be a square, a rectangle or a parallelogram.
6. Draw a pyramid with a triangular base.
7. Consider a rectangular prism.
   (a) On which pairs of faces are there congruent rectangles?
   *(b) Describe the positions of 3 planes of symmetry of a rectangular prism. Think of a chalk box and 3 different planes each of which would divide the box into two parts which are congruent to each other. Also see Problem 8 of Section 4-3.
8. Consider a triangular prism of which one face is an equilateral triangle.
   (a) Describe the congruent triangles or rectangles.
   *(b) Describe four planes of symmetry.

*9. Answer the questions in Problem 8 if the triangular faces of a triangular prism are scalene triangles.

10. Are there any congruent triangles or polygons on the faces of a pyramid with a square base?

11. Draw a picture of a book showing two pages at different angles.

12. Draw a picture of a flat target with an arrow through it.

Chapter 5

RELATIVE ERROR

5-1. Greatest Possible Error

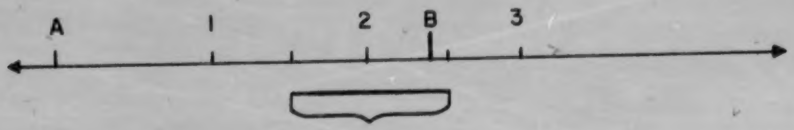
The process of measurement plays such an important part in contemporary life that everyone should have a clear understanding of its nature. A substantial part of the arithmetic taught in the elementary school relates to measurement. Most of the early work in measurement is designed to familiarize you with common units of measurement and their use, and with the ratios between their measures.

The basic concept to be developed in this chapter is that the process of measurement of a single thing yields a number which represents the approximate number of units. This is in contrast to the process of counting separate objects, which yields an exact number. When the number of separate objects is rounded or estimated, the resulting number is treated as an approximation in the same sense as a measurement. Since measurements are approximate, calculations made with their measures, such as sums or products, yield results which are also approximate.

When you use numbers to count separate objects you need only counting numbers. In counting you set up a one-to-one correspondence between the objects counted and the members of the set of counting numbers. When you count the number of people in a classroom you know the result will be a counting number; there may be exactly 11, but there cannot be $11\frac{1}{4}$ or $10\frac{1}{2}$. If there are a great many people, or if you are not sure you have counted correctly, you may say there are "about 300," rounding the number to the nearest hundred. However, if you did count carefully you could determine exactly the number of people in the room.

When you measure something, the situation is different. When you have measured the length of a line segment with a ruler divided into quarter-inches, the end of the segment probably fell between two quarter-inch marks, and you had to judge which mark appeared closer. Even if the end seemed to fall almost exactly on a quarter-inch mark, if you had looked at it through a magnifying glass you would probably have found that there was a difference. And if you had then changed to a ruler with the inches divided into sixteenths, you might have decided that the end of the segment was nearer to one of the new sixteenth-inch marks than to a quarter-inch mark.

In any discussion of measurement we assume proper use of instruments. Improper use of instruments can occur through ignorance, or carelessness. These mistakes can be corrected by learning how to use the instrument and by careful inspection during the measurement process. But, even with the best instruments and techniques, scientists agree that measurement cannot be considered exact, but only approximate. The important thing to know is just how inexact a measurement may be, and to state clearly how inexact it may be.



Look at the line above, which pictures a scale divided into one-inch units, (not drawn to scale). The zero point is labeled "A", and point B is between the 2-inch mark and the 3-inch mark. Since B is clearly closer to the two-inch mark, we may say that the measurement of segment $\overline{AB}$ is 2 inches. However, any point which is more than $1\frac{1}{2}$ inches from A and less than $2\frac{1}{2}$ inches from A would be the endpoint of a segment whose length, to the nearest inch, is also 2 inches. The mark below the line shows the range within which the endpoint of a line segment 2 inches long (to the nearest inch) might fall. The length of such a segment might be almost $\frac{1}{2}$ inch less than 2 inches, or almost $\frac{1}{2}$ inch more than 2 inches. We therefore say that, when a line segment is measured to the nearest inch, the "greatest possible error" is $\frac{1}{2}$ inch. This does not mean that you have made a mistake (or that you have not). It simply means that, if you measure properly to the nearest inch, any measurement more than $1\frac{1}{2}$ inches and less than $2\frac{1}{2}$ inches will be correctly reported in the same way, as 2 inches. Consequently such measurements are sometimes stated as $(2 \pm \frac{1}{2})$ inches. (The symbol "±" is read "plus or minus".) By this we mean that the greatest possible error in the measurement is $\frac{1}{2}$ inch. To say it in another way, the measurement 2 inches is correct to the nearest inch.

Suppose we see a sign at a milepost which says "Chicago 73 miles." What unit of measurement are we to assume and what error possible? We do not really know, although a reasonable interpretation is that the mileage is correct to the nearest mile, and that an error of no more than $\frac{1}{2}$ mile is to be expected.

But what about the milepost which indicates a distance of 1 mile to the next milepost? Are we to assume that the unit is 1 mile and that this measured mile indicates a distance lying between 0.5 miles and 1.5 miles? Clearly, this is not a reasonable interpretation in this case, since we expect this measurement to be much more precise.

To state what the greatest possible error in a measurement is, we need to know how the measurement was made and how accurate the instrument of measurement was. Ordinarily, we do not know this background. In fact, we really do not know just what is meant when someone says, an object is "two inches long," or "the distance is 1 mile." Usually it does not particularly matter whether the 2 inch measurement is correct to $\frac{1}{16}$" or to $\frac{1}{32}$". or the mile measurement is correct to .1 mile or .01 mile. Sometimes, though, it is important to indicate what the error may be. In scientific and technical work the greatest possible error is specifically stated. For example, a length would be stated as $(2 \pm \frac{1}{16})$ inches or possibly (4 ± 0.005) inches, not simply as 2 inches or 4 inches.

Often in business and industry the term "tolerance" is used. By tolerance we mean the greatest error which is allowed. The tolerance might be set by the person who is purchasing a certain manufactured product or by the operation of a machine. For instance, an automobile manufacturer might specify that the cylinders of an engine should have a diameter of 5 inches with a tolerance of one-thousandth of an inch. This means the diameter cannot vary more than 0.001 inch from 5 inches; the dimension would be given as (5 ± 0.001) inches. On the other hand, a producer of water pumps might demand a tolerance different from 0.001 inch. Laws often specify tolerance for instruments in commercial use like scales for weighing. Scales are allowed to vary within certain limits. Court cases are sometimes decided on the basis of tolerances allowed in the calibration of police car speedometers.

Class Exercises 5-1

1. When you measure to the nearest $\frac{1}{4}$ inch, what is the greatest possible error?
2. A meter stick is divided into centimeters and tenths of a centimeter (millimeters). A line segment was measured with such a scale and stated to be 3.7 cm.
   (a) What was the unit of measurement?
   (b) The measurement is (3.7 ± ?) cm.
   (c) What was the greatest possible error?
   Give your answer in cm. and also in mm.
3. Scientists frequently measure to the nearest $\frac{1}{100}$ of a centimeter. The greatest possible error for such a unit is ______ cm. or ______ mm.
4. The greatest possible error in a measurement is always what fractional part of the unit used?
5. A tolerance of .0005 in. is specified for a metal sheet of thickness 0.350 inches. Allowable thickness for the sheet will be between ______ in. and ______ in.

5-2. Precision and Significant Digits

Consider the two measurements, $10\frac{1}{8}$ inches and $12\frac{1}{2}$ inches. As commonly used, these measurements do not indicate what unit of measurement was used. Suppose that the unit for the first measurement is $\frac{1}{8}$ inch, and the unit for the second measurement is $\frac{1}{2}$ inch. Then we say that the first measurement is more precise than the second, or has greater precision. Notice that the precision of a measurement depends upon the smallest unit used in the measurement. The greatest possible error of the first measurement is $\frac{1}{2}$ of $\frac{1}{8}$ inch, or $\frac{1}{16}$ inch, and of the second is $\frac{1}{2}$ of $\frac{1}{2}$ inch, or $\frac{1}{4}$ inch. The greatest possible error is less for the first measurement than for the second measurement. Hence the more precise of two measurements is the one made with the smaller unit, and for which the greatest possible error is therefore the smaller.

To summarize: The greatest possible error in a measurement is $\frac{1}{2}$ of the smallest unit of measure used in the measurement. The more precise of two measurements is the one for which the greatest possible error is smaller.

For example, a measurement made using a metric scale may use centimeter and millimeter divisions (units) of the scale. A measurement of 37.6 cm., made to the nearest .1 cm. (or nearest mm.) has a greatest possible error of $\frac{1}{2}$ (1 mm.) or $\frac{1}{2}$ (.1 cm.) = .05 cm. This says that the actual length lies between (37.6 - .05) cm. and (37.6 + .05) cm. We indicate these limitations by writing $37.6 \pm .05$. This is a convenient way of saying two things at once, by use of the ± symbol.

It is very important that measurements be stated in a way which shows clearly how precise they are. When we use the ± notation there is no doubt about what we mean. Another way to show clearly what we mean is to make certain agreements about what is meant when we write a number in decimal form--the form which most often occurs in scientific and technical measurement. When we write that a length has been measured as 17.62 inches we understand that the measurement has been made with an error no greater than .005 in. Thus the measure 17.62 is correct to the second decimal place to the right of the decimal point. In the ± notation this would be equivalent to writing (17.62 ± .005) inches for the measurement. By this agreement, each of the four digits in 17.62 serves a real purpose, or is "significant."

In measures like 1462, 3.1 and .29637 all the digits are understood to be significant. But in a numeral like 0.008 the three zeros simply serve to fix the decimal point. In this case we say only the 8 is a significant digit..

In the numeral 2.008, all four digits (2, 0, 0, 8) are significant. In a numeral like 0.0207 the first two zeros are *not* significant but the third is. Thus, 0.0207 has three significant digits 2, 0, 7.

When we write 2960 ft. or 93,000,000 miles it is not clear which, if any, of the zeros are significant. We agree that they are *not* significant, since they serve to fix the location of the decimal point. Thus 2960 ft. has three significant digits (2, 9, 6) in its measure. The measurement is precise to the nearest 10 ft. and the greatest possible error is 5 ft.

Whenever we wish any of the zeros at the end of a number like 28000 or 2960 to be significant, we agree to indicate the final zero which is significant. Thus $296\underline{0}$ ft. indicates a measurement correct to the foot. The measure has four significant digits (2, 9, 6, 0). The measurement $93,\underline{0}00,000$ miles is correct to the nearest **100,000** miles. The numeral has **three** significant digits 9, 3, 0.

Definition. *A digit in a decimal numeral is said to be a "significant digit" if it serves a purpose other than simply to locate (or emphasize) the decimal point*.

Some further examples:

Numeral	*Significant Digits* (*in order*)
39060	3, 9, 0, 6
73.40	7, 3, 4, 0
692	6, 9, 2

0.00523	5, 2, 3
8.0057	8, 0, 0, 5, 7

In 39060, the "0" between 9 and 6 is significant, the other "0" is not; it simply locates the decimal point (understood). In the numeral 73.40, the "0" is significant because it is not necessary to have it to locate the decimal point. In 0.00523, all the "0's" are used simply to locate or emphasize the decimal point. We understand that the left-most zero may or may not be written, and, if written, is simply for clarity in locating the decimal point and reading the number.

When a number is written in scientific notation we agree that all of the digits in the first factor are significant, thus:

73,000 ft. $= 7.3 \times 10^4$ ft.
$73,\underline{0}00$ ft. $= 7.30 \times 10^4$ ft.
$73,00\underline{0}$ ft. $= 7.3000 \times 10^4$ ft.

Also, the measurement 2.99776×10^{10} cm./sec., for the velocity of light, has 6 significant digits; the measurement 2.57×10^{-9} cm. for the radius of the hydrogen atom, has 3 significant digits; the measurement for the national debt in 1957, 2.8×10^{11} dollars, has 2 significant digits; 4.800×10^8 has 4 significant digits. In the last case, the two final zeros are significant. Were they not, the number should have been written as 4.8×10^8. It is this possiblity of indicating significant digits in scientific notation which is one of the principal advantages of the notation.

Exercises 5-2

1. Suppose you measured a line to the nearest hundredth of an inch. Which of the following states the measurement best?

   3.2 inches   3.20 inches   3.200 inches

2. Suppose you measured to the nearest tenth of an inch. Which of the following could you use to state the result?

   4 inches   4.0 inches   4.00 inches   (4.0 ± 0.05) inches

3. Tell which measurement in each pair has the greater precision.
   (a) 5.2 feet, $(2\frac{1}{4} \pm \frac{1}{8})$ feet
   (b) 0.68 feet, $(23.5 \pm .05)$ feet
   (c) 0.235 inches, 0.146 inches

4. What is your age to the nearest year--that is, at your nearest birthday will you be ten, eleven, twelve, thirteen, ... ? All of you who say "13" must be between _______ and $13\frac{1}{2}$ years old.

5. (A) For each measurement below tell the place value of the last significant digit.
   (B) Tell the greatest possible error of the measurements.
   (a) 52700 feet
   (b) $527\underline{0}$ feet
   (c) $527\underline{0}0$ feet
   (d) 52.7 feet
   (e) 0.5270 feet
   (f) 527.0 feet

6. (a) Which of the measurements in Problem 5 is the most precise?
   (b) Which is the least precise?
   (c) Do any two measurements have the same precision?

7. Show by underlining a zero the precision of the following measurements:
   (a) 4200 feet measured to the nearest foot
   (b) 23,000 miles, measured to the nearest hundred miles
   (c) 48,000,000 people, reported to the nearest ten-thousand

8. Tell the number of significant digits in each measurement:
   (a) 520 feet
   (b) 32.46 in.
   (c) 0.002 in.
   (d) 403.6 ft.
   (e) $25{,}8\underline{0}0$ ft.
   (f) 0.0015 in.
   (g) 38.90 ft.
   (h) 0.0603 in.

9. How many significant digits are in each of the following:
   (a) 4.700×10^5
   (b) 4.700×10^4
   (c) 4.7×10^{15}
   (d) 6.70×10^{-4}
   (e) 4.7000×10
   (f) 2.8×10^9

5-3. Relative Error, Accuracy, and Percent of Error

While two measurements may be made with the same precision (that is, with the same unit) and therefore with the same greatest possible error, this error is more important in some cases than in others. An error of $\frac{1}{2}$ inch in measuring your height would not be very misleading, but an error of $\frac{1}{2}$ inch in measuring your nose would be misleading. We can get a measure of the importance of the greatest possible error by comparing it with the measurement. Consider these measurements and their greatest possible errors:

4 in. ± 0.5 in.   58 in. ± 0.5 in.

Since these measurements are both made to the nearest inch, the greatest possible error in each case is 0.5 inch. If we divide the measure of the greatest possible error by the number of units in the measurement we get these results. (Note that the measures are numbers and the measurements are not. We shall refer to the number of units in the measurement as the measure.)

$$\frac{0.5}{4} = \frac{5}{40} = 0.125$$

$$\frac{0.5}{58} = \frac{5}{580} \approx 0.0086$$

The quotients 0.125 and 0.0086 are called relative errors. The relative error of a measurement is defined as the quotient of the measure of the greatest possible error by the measure.

$$\text{Relative error} = \frac{\text{measure of the greatest possible error}}{\text{the measure}}$$

Percent of error is the relative error expressed as a percent. In the above two examples the relative error expressed as a percent is 12.5% and 0.86% . When written in this form it is called the percent of error.

The measurement with a relative error of 0.0086 (0.86%) is more accurate than the measurement with a relative error of 0.125 (12.5%). By definition a measurement with a smaller relative error is said to be more accurate than one with a larger relative error.

The terms accuracy and precision are used in industrial and scientific work in a special technical sense even though they are often used loosely and as synonyms in everyday conversation. Precision depends upon the size of the unit of measurement, which is twice the greatest possible error, while accuracy is the relative error or percent of error. For example, 12.5 pounds and 360.7 pounds are equally precise; that is, precise to the nearest 0.1 of a pound (greatest possible error in each case is 0.05 pound). The two measurements do not possess the same accuracy. The second measurement is more accurate. You should verify the last statement by computing the relative errors in each case and comparing them.

An astronomer, for example, making a measurement of the distance to a galaxy may have an error of a trillion miles (1,000,000,000,000 miles) yet be far more accurate than a machinist measuring the diameter of a steel pin to the nearest 0.001 inch.

Again, a measurement indicated as 3.5 inches and another as 3.5 feet are equally accurate but the first measurement is probably more precise. Why?

Suppose we have two measurements of the same quantity, say 3.5 in. and 3.500 in. There are two significant digits in the first measure and four significant digits in the second measure. What does the number of significant digits in the measure tell us about the accuracy (relative error) of the measurement. Clearly, the greater the number of significant digits in the measure, the greater the accuracy of the measure. To illustrate this we write the following:

3.5 in.	3.500 in.
two significant digits (3,5)	four significant digits (3,5,0,0)
Relative error = $\frac{.05}{3.5}$	Relative error = $\frac{.0005}{3.500}$
or	or
Accuracy ≈ .01	Accuracy ≈ .0001
Precision = .1 in.	Precision = .001 in.

Try a similar comparison for the two measurements 93,000,000 miles and (.03 ± .005) in.

93,000,000 miles	(.03 ± .005) in.
Two significant digits [9,3]	One significant digit [3]
Relative error = $\frac{500,000}{93,000,000}$	Relative error = $\frac{.005}{.03}$
or	or
Accuracy ≈ .005	Accuracy ≈ .2
Precision 1,000,000 mi.	Precision .01 in.

Exercises 5-3

In all computation express your answer so that it includes two significant digits.

1. State the greatest possible error for each of these measurements.
   (a) (52 ± 0.5) ft.
   (b) (4.1 ± 0.05) in.
   (c) 2580 mi.
   (d) $36\underline{0}$ ft.
   (e) 7.03 in.
   (f) 0.006 ft.
   (g) 5.4×10^4 mi.
   (h) $54,0\underline{00}$ mi.

2. Find the relative error of each measurement in Problem 1.

3. Find the greatest possible error and the percent of error for each of the following measurements.
   (a) (9.3 ± 0.05) ft.
   (b) 0.093 ft.
   (c) 9.30×10^2 ft.
   (d) 9.30×10^4 ft.

4. What do you observe about your answers for Problem 3? Can you explain why the percents of error should be the same for all of these measurements?

5. Find the precision of the following measurements.
   (a) 26.3 ft.
   (b) 0.263 ft.
   (c) 2630 ft.
   (d) 51,000 mi.
   (e) 5.1 ft.
   (f) 0.051 in.

6. How many significant digits are there in each of the following?
   (a) 52.1 in.
   (b) 52.10 in.
   (c) 3.68 in.
   (d) 368.0 in.

7. Find the relative error of each of the measurements in Problem 6.

8. From your answers for Problems 6 and 7, can you see any relation between the number of significant digits in a measure and the relative error in the measurement? What is the relation between the number of significant digits in a measure and the accuracy of the measurement?

9. Without computing, can you tell which of the measurements below has the greatest accuracy? Which is the least accurate?
23.6 in. 0.043 in. 7812 in. 0.2 in.

10. Arrange the following measurements in the order of their precision (from least to greatest):
(a) $(36\frac{1}{2} \pm \frac{1}{4})$ in., $(27 \pm \frac{1}{32})$ in., $(32\frac{3}{8} \pm \frac{1}{16})$ in.,
$(46\frac{2}{7} \pm \frac{1}{14})$ in., $(22.25 \pm .125)$ in.
(b) 4.62 in., 3.041 in., 3 in., 82.4 in., 0.3762 in.

*11. Arrange the following measurements in order of their accuracy (from least to greatest):
$(6 \pm \frac{1}{2})$ ft. (3.2 ± 0.005) in. (7.2 ± 0.05) miles
$(3\frac{1}{2} \pm \frac{1}{8})$ in. 3 yd. $(4 \pm \frac{1}{4})$ in. .

12. Count the number of significant digits in each of the following measures:
(a) 43.26
(b) 4,607
(c) 32.004
(d) 0.0062
(e) 0.6070
(f) 0.0030
(g) 4.0030
(h) 0.03624
(i) $76{,}00\underline{0}$
(j) 43,000
(k) 0.036
(l) 200.00004.

13. Express the following measures in scientific notation:
(a) 463,000,000
(b) $327{,}\underline{0}00$
(c) 0.000462
(d) 32.004
(e) 2
(f) 0.0000400
(g) 36.8×10^5
(h) 0.80×10^{-7}
(i) 72 billion.

14. By inspection arrange the following numbers in order of their magnitude, from least to greatest. List by letter only.

(a) 3.6×10^{5}
(b) 3.5×10^{8}
(c) 4×10^{-6}
(d) 3.527×10^{8}
(e) 3.5×10^{-12}
(f) 4.1×10^{6}
(g) 3.527×10^{2}
(h) 3.55×10^{8}
(i) 3.4×10^{-7}
(j) 3.39×10^{-8}

15. BRAINBUSTER. A master machinist measures a $3\frac{1}{2}$ inch piston head to the nearest 0.0001 inch while an astronomer measures by the parallax, the distance to Canis Major (the star Sirius) correct to the nearest 10,000,000 miles. The distance to Sirius is 8.6 light years (1 light year $\approx 6 \times 10^{12}$ miles). Which measurement is more accurate?

5-4. Adding and Subtracting Measures

Since measurements are never exact, the answers to any questions which depend on those measurements are also approximate. For instance, suppose you measured the length of a room by making two marks on a wall, which you called A and B, and then measuring the distances from the corner to A, from A to B, and from B to the other corner. Measurements such as these whose measures are to be added, should all be made with the same precision. Suppose, to the nearest fourth of an inch, the measurements were $72\frac{1}{4}$ inches, $40\frac{2}{4}$ inches, $22\frac{3}{4}$ inches. You would add the measures to get $135\frac{2}{4}$. Therefore the measurement is $135\frac{2}{4}$ inches. Of course, the distances might have been shorter in each case. The measures could have been almost as small as $72\frac{1}{8}$, $40\frac{3}{8}$, and $22\frac{5}{8}$ in which case the distance would have been almost as small as $135\frac{1}{8}$ inches, which is three-eighths of an inch less than $135\frac{2}{4}$ inches. Also, each distance might have been longer by nearly one-eighth of an inch, in which case the total length might have been almost three-eighths of an inch longer than $135\frac{2}{4}$ inches. *The greatest possible error of a sum is the sum of the greatest possible errors.* If we were adding measures of 37.6, 3.5, and 178.6, the greatest possible error of the sum would be .05 + .05 + .05 or .15. The result of this addition would be shown as 219.7 ± .15.

Computation involving measures is very important in today's world. Many rules have been laid down giving the accuracy or precision of the results obtained from computation with approximate measures. Too many rules, however, might create confusion and would never replace basic knowledge of approximate data. If the meaning of greatest possible error and of relative error is understood, the precision and accuracy of the result of computation with approximate data can usually be found by applying common sense and judgment. Common sense would tell us that with a large number of measurements the errors will, to a certain extent, cancel each other.

The general principle is that the sum or difference of measures cannot be more precise than the least precise measure involved. Therefore to add or subtract numbers arising from approximations, first round each number to the unit of the least precise number and then perform the operation.

As we have seen, the greatest possible error of a sum (or difference) of several measures is the sum of the greatest possible errors of the measures involved. (To estimate the *expected* error of a sum, taking into account the way the errors would often cancel each other, we need to use some ideas of probablility, not yet at our disposal.)

Exercises 5-4

1. Find the greatest possible error for the sums of the measurements in each of the following. (When a measurement is given as $5\frac{1}{2}$ in., you may assume that the unit of measurement was $\frac{1}{2}$ in. and correspondingly for other fractions.)

(a) $5\frac{1}{2}$ in., $6\frac{1}{2}$ in., $3\frac{0}{2}$ in.

(b) $3\frac{1}{4}$ in., $6\frac{1}{2}$ in., 3 in.

(c) 4.2 in., 5.03 in.

(d) 42.5 in., 36.0 in., 49.8 in.

(e) 0.004 in., 2.1 in., 6.135 in.

(f) $2\frac{3}{4}$ in., $1\frac{5}{16}$ in., $3\frac{3}{8}$ in.

2. Add the following measures:

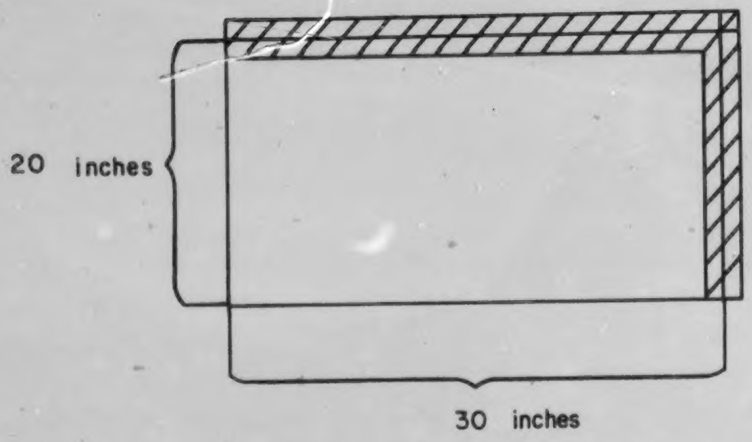
(a) 42.36, 578.1, 73.4, 37.285, 0.62
(b) 85.42, 7.301, 16.015, 36.4
(c) 9.36, 0.345, 1713.06, 35.27

3. Subtract the following measures:

(a) 7.3 - 6.28
(b) 735 - 0.73
(c) 5430 - 647

5-5. Multiplying and Dividing Measures

You know that the number of units in the area of a rectangle is found by multiplying the number of units in the length by the number of the same units in the width. Suppose that the dimensions of a rectangle are 30 inches and 20 inches. Since the measuring was done to the nearest inch, the measures can be stated as $(30 \pm .5)$ and $(20 \pm .5)$. This means that the length might be almost as small as 29.5 inches and the width almost as small as 19.5 inches. The length might be almost as large as 30.5 inches and the width almost as large as 20.5 inches.



Look at the sketch to see what this means. The outside lines show how the rectangle would look if the dimensions were as large as possible. The inner lines show how it would look if the length and width were as small as possible. The shaded part shows the difference between the largest possible area and the smallest possible area with the given measurements.

Let us see what the differences are. The given measurements are 20 ft. × 30 ft. The smallest possible dimensions are (20 - .5) ft. × (30 - .5) ft. and the largest possible dimensions are (20 + .5) ft. × (30 + .5) ft.

Least Possible Area	Given Area	Greatest Possible Area
(20 -.5)×(30 -.5) sq. ft.	20 × 30 sq.ft.	(20 +.5)×(30 +.5) sq.ft.
or		or
(600 - 10 - 15 +.25)sq.ft.	or	(600 + 10 + 15 +.25)sq.ft.
or		or
575.25 sq. ft.	600 sq. ft.	625.25 sq. ft.

Hence there is a difference of (625.25 - 575.25) or 50 sq. ft. in the two possible errors. The computed area of 20 × 30 sq. ft. or 600 sq. ft. is about 25 sq. ft. greater than the smallest area and about 25 sq. ft. less than the largest area possible.

Therefore, if we wish to be very careful about our statements we must make clear what is meant when we say the area of the rectangle is 600 sq. ft. As we have seen, this answer is not correct to 1 square foot, but it is correct to less than 100 sq. ft. If we wish to indicate the situation as best as we know it we may write the area as (600 ± 25) sq. ft. (We have chosen here to round the greatest possible area 625.25 to 625 sq. ft. You might prefer to write 575.25 sq. ft. $\leq$ the area $\leq$ 625.25 sq. ft.) If we choose to write the area as 600 sq. ft. we must interpret the numeral as one with one significant digit. This says the area is given to within 100 sq. ft. and hence lies between 550 sq. ft. and 650 sq. ft. This is correct, but not quite so good a result as our answer (600 ± 25 sq. ft.)

It is really impossible to give a satisfactory rule for the multiplication of approximate measures in decimal or fractional form. However, when data are expressed in decimal form a rough guide can be suggested for finding a satisfactory product. The number of significant digits in the product of two numbers is not more than the number of significant digits in the less accurate factor.

Note that this says the number of significant digits is not more than the number of significant digits in the less accurate factor--it does not assure you that there will be that many!

As an illustration of this principle consider the following problem: What is the area of a rectangle with sides measured as 10.4 cm. and 4.7 cm.?

To find the area we might multiply 10.4 by 4.7 to obtain 48.88. Now there are three significant digits in 10.4 and only two significant digits in 4.7 Hence, the product cannot have more than two significant digits and we round the area to 49 sq. cm. Hence, the area of the rectangle is $\approx$ 49 sq. cm.

If we wish to find a better estimate of the possible error we must use the "±" scheme we used earlier. Since

$$\begin{aligned}(10.4 + .05)(4.7 + .05) &= 48.88 + .52 + .235 + .0025 \\ &= 49.6375\end{aligned}$$

and

$$(10.4 - .05)(4.7 - .05) = 48.88 - .52 - .235 + .0025$$
$$= 48.1275,$$

we see that 48.1 sq. cm. < the area of the rectangle < 49.7. If we use only two significant figures we see that the area lies between 48 and 50 sq. cm. Hence (49 ± 1) sq. cm. is a good answer.

You might ask, why not round the numeral 10.4 to 10 and work only with two significant digits in each factor? Then we would get

$$10 \times 4.7 \text{ sq. cm.} = 47 \text{ sq. cm.}$$

for the area, and we see that this is not correct to two significant figures.

For such reasons as this, we ordinarily agree to the following general procedure when multiplying two factors which do not have the same number of significant digits.

If one of the two factors contains more significant digits than the other, round off the factor which has more significant digits so that it contains only one more significant digit than the other factor.

Suppose we wish to find the circumference of a circle with the diameter D equal to 5.1 mm. The circumference $C = \pi D$. What value of π shall we use? Since the diameter 5.1 is given to two significant digits we use three significant digits for π or $\pi \approx 3.14$. Then

$$C = \pi D \approx 3.14 \times 5.1 = 16.014,$$

which we round to 16, since only two digits are significant in the product. Hence, the circumference is approximately 16 mm.

If we were dealing with a large circle with diameter D measured as 1012 inches, then we would use $\pi \approx 3.1416$ and round the result of the multiplication $C = (3.1416)(1012)$ to four significant digits.

Division is defined by means of multiplication. Therefore it is reasonable to follow the procedure used for multiplication in doing divisions involving approximate data.

When a multiplication or division involves an exact number such as 2 in the formula for the circumference of a circle ($C = 2\pi r$), the approximate number determines the number of significant digits in the answer. We ignore the exact number in determining the significant digits in the answer. An exact number is a number that is not found by measuring.

Exercises 5-5

1. Suppose a rectangle is $2\frac{1}{2}$ inches long and $1\frac{1}{2}$ inches wide. Make a drawing of the rectangle. Show on the drawing that the length is $(2\frac{1}{2} \pm \frac{1}{4})$ inches and the width $(1\frac{1}{2} \pm \frac{1}{4})$ inches. Then find the largest area possible and the smallest area possible, and find the difference, or uncertain part. Then find the area with the measured dimension, and find the result to the nearest $\frac{1}{2}$ square inch.

2. Multiply the following approximate numbers:
   (a) 4.1×36.9
   (b) 3.6×4673
   (c) $3.76 \times (2.9 \times 10^4)$

3. Divide the following approximate numbers:
   (a) $3.632 \div 0.83$
   (b) $0.000344 \div 0.000301$
   (c) $(3.14 \times 10^6) \div 8.006$

4. Find the area of a rectangular field which is 835.5 rods long and 305 rods wide.

5. The circumference of a circle is stated $C = \pi d$, in which d is the diameter of the circle. If π is given as 3.141593, find the circumferences of circles whose diameters have the following measurements:
   (a) 3.5 in.

(b) 46.36 ft.
(c) 6 miles.

6. A machine stamps out parts each weighing 0.625 lb. How much weight is there in 75 of these parts?

7. Assuming that water weighs 62.5 lb. per cu. ft., what is the volume of 15,610 lbs.?

There are many rough rules for computing with approximate data but they have to be used with a great deal of common sense. They don't work in all cases. The modern high speed computing machine, which adds or multiplies thousands of numbers per second, has to have special rules applied to the data which are fed to it. Errors involved in rounding numbers add up or disappear in a very unpredictable fashion in these machines. As a matter of fact "error theory" as applied to computers is an active field of research today for mathematicians.

Chapter 6

REAL NUMBERS

6-1. Review of Rational Numbers

In your study of mathematics you have used several number systems. You began with the counting numbers, and you may have known a good deal about these numbers before you entered the first grade in school. These numbers are so familiar that it is easy to overlook some of the ways in which the system of counting numbers differs from other systems. Consider the following questions:

(a) Think of a particular counting number. What is the next smaller counting number? the next larger? If n represents a counting number, what represents the next smaller counting number? the next larger?

(b) Is there a counting number which cannot be used as a replacement for n in your answer to question (a)? Why?

(c) Is there a smallest counting number? a largest? If so, what are they?

(d) Is the set of counting numbers closed under the operation of

(1) addition?
(2) subtraction?
(3) multiplication?
(4) division?

(e) How many counting numbers are there between 8 and 11? between 3002 and 4002? between 168 and 169? Between any two counting numbers is there always another counting number?

In Chapter 1 you studied positive and negative rational numbers. The set of integers contains the set of counting numbers (called positive integers). For each positive integer a there is an opposite number ^-a. The opposites of positive integers are called negative integers. If a is a counting number, then $a + {}^-a = 0$. What integer is neither positive nor negative?

The set of integers is contained in another set of numbers which we call the set of rational numbers. As you know, the set of integers is adequate for many purposes, such as reporting the population of a country, the number of dollars you have (or owe), the number of vertices in a triangle, and so on. The integers alone are not suitable for many purposes, particularly for the process of measurement. If we had only the integers to use for measuring we would have to invent names for subdivisions of units. We do this to some extent; instead of saying $5\frac{1}{3}$ feet we sometimes say 5 feet 4 inches. But we do not use a different name for a subdivision of an inch. Instead, we speak of $7\frac{1}{4}$ inches, or 7.25 inches, using rational numbers which are not integers. If we had only the integers, we could never speak of $3\frac{1}{2}$ quarts, or 2.3 miles, or 0.001 inch.

Recall that a rational number may be named by the fraction symbol "$\frac{p}{q}$", where p and q are integers, and $q \neq 0$.

Just as there is a negative integer which corresponds to each positive integer (or counting number), there is a negative rational number which corresponds to each positive rational number.

You already may have observed the familiar properties for rational numbers, which may be summarized as follows:

Closure: If a and b are rational numbers, then $a + b$ is a rational number, $a \cdot b$ (more commonly written ab) is a rational number, $a - b$ is a rational number, and $\frac{a}{b}$ is a rational number if $b \neq 0$.

Commutativity: If a and b are rational numbers, then $a + b = b + a$, and $a \cdot b = b \cdot a$, $(ab = ba)$.

Associativity: If a, b, and c are rational numbers, then $a + (b + c) = (a + b) + c$, and $a(bc) = (ab)c$.

Identities: There is a rational number zero such that if a is a rational number, then $a + 0 = a$. There is a rational number 1 such that $a \cdot 1 = a$.

Distributivity: If a, b, and c are rational numbers, then $a(b + c) = ab + ac$.

Additive inverse: If a is a rational number, then there is a **rational number** $({}^-a)$ **such that** $a + ({}^-a) = 0$.

Multiplicative inverse: If a is a rational number and $a \neq 0$, then there is a **rational number** b **such that** $ab = 1$.

Order: If a and b are different rational numbers, then either $a > b$, or $a < b$.

Class Exercise 6-1

1. Is there a smallest negative integer? A largest one?

2. If n represents a negative integer, what represents the next larger one? the next smaller one?

3. Is the set of negative integers closed under the operation of

   (a) addition?

   (b) subtraction?

   (c) multiplication?

   (d) division?

4. Express each of the following in the form $\frac{p}{q}$ or ${}^-(\frac{p}{q})$, where p and q are counting numbers.

   (a) $5\frac{3}{4}$

   (b) $7\frac{1}{8}$

   (c) 12

   (d) 0.47

   (e) $\frac{5}{{}^-3}$

   (f) $6 + \frac{9}{10}$

   (g) 3.7

   (h) ${}^-7 + \frac{1}{3}$

5. Which of the properties for rational numbers does each statement illustrate?

   (a) $\frac{2}{3} + 0 = \frac{2}{3}$

   (b) ${}^-(\frac{1}{8}) + \frac{3}{4} = \frac{5}{8}$ and $\frac{5}{8}$ is a rational number.

   (c) ${}^-(\frac{2}{3}) \cdot {}^-(\frac{5}{8}) = {}^-(\frac{5}{8}) \cdot {}^-(\frac{2}{3})$

(d) $\frac{1}{2} \cdot (\frac{3}{5} + \frac{7}{10}) = (\frac{1}{2} \cdot \frac{3}{5}) + (\frac{1}{2} \cdot \frac{7}{10})$

(e) $1 \cdot {}^{-}(\frac{7}{8}) = {}^{-}(\frac{7}{8})$

(f) $\frac{17}{10} + (\frac{1}{10} + \frac{9}{10}) = (\frac{17}{10} + \frac{1}{10}) + \frac{9}{10}$

(g) ${}^{-}(\frac{1}{5}) \cdot \frac{2}{3} = {}^{-}(\frac{2}{15})$ and ${}^{-}(\frac{2}{15})$ is a rational number.

6. What is the additive inverse of ${}^{-}(\frac{7}{4})$?
7. What is the multiplicative inverse of ${}^{-}(\frac{7}{4})$?
8. What is another name for "multiplicative inverse"?
9. If $\frac{p}{q}$, or ${}^{-}(\frac{p}{q})$, is the simplest fractional form for a rational number which is an integer, what must q be?
10. How can you tell whether two fractions represent the same rational number?
11. What are three other names for the rational number $\frac{5}{7}$?

Exercises 6-1

1. Look at each statement below and tell which of the properties listed for rational numbers it illustrates.

   (a) ${}^{-}(\frac{3}{4}) + \frac{5}{6} = \frac{1}{12}$, and $\frac{1}{12}$ is a rational number.

   (b) $\frac{5}{8} + 0 = \frac{5}{8}$

   (c) $1 \cdot {}^{-}(\frac{3}{4}) = {}^{-}(\frac{3}{4})$

   (d) ${}^{-}(\frac{3}{4}) \cdot {}^{-}(\frac{5}{8}) = {}^{+}\frac{15}{32}$ and ${}^{+}\frac{15}{32}$ is a rational number.

   (e) $\frac{2}{3} \cdot (\frac{1}{3} + \frac{1}{2}) = (\frac{2}{3} \cdot \frac{1}{3}) + (\frac{2}{3} \cdot \frac{1}{2})$

   (f) ${}^{-}(\frac{5}{8}) \cdot {}^{-}(\frac{1}{3}) = {}^{-}(\frac{1}{3}) \cdot {}^{-}(\frac{5}{8})$

   (g) $\frac{11}{10} + (\frac{3}{10} + \frac{7}{10}) = (\frac{11}{10} + \frac{3}{10}) + \frac{7}{10}$

2. Express each of the following in the form $\frac{p}{q}$ or ${}^-(\frac{p}{q})$, where p and q are counting numbers.

   (a) $17\frac{1}{2}$

   (b) $\frac{5}{{}^-7}$

   (c) ${}^-4 + \frac{1}{3}$

   (d) ${}^-0.35$

   (e) 10

   (f) 17.03

3. Write each of these in simplest fractional form.

   (a) $\frac{14}{28}$

   (b) $\frac{18}{24}$

   (c) ${}^-0.62$

   (d) $6\frac{2}{7}$

   (e) ${}^-(8\frac{1}{4})$

   (f) 12.5

4. What is the additive inverse of each of the following?

   (a) ${}^-28$

   (b) 756

   (c) ${}^+3\frac{1}{7}$

   (d) ${}^-(\frac{176}{5})$

5. Complete the statement, "The simplest name for a rational number written in the form $\frac{a}{b}$ is the one in which a and b have no common factor except ____________."

6. A rational number which does not have a reciprocal is the number $\frac{p}{q}$ when p is ____________.

7. Arrange the following rational numbers in order. List the greatest one last.

$$\frac{4}{7},\ \frac{3}{8},\ 0.41,\ \frac{7}{16},\ \frac{2}{5},\ {}^-4,\ {}^-(\frac{2}{3}),\ 0$$

*8. Find the average of the two rational numbers ${}^-8$ and ${}^+4$.

*9. Is it always possible to find the average of two integers and have the average be an integer? Explain.

10. Multiply each of the following by 10.
    (a) 0.33333
    (b) 0.090909
    (c) 16.31212
    (d) 0.142142
    (e) 13.46333
    (f) 846.4646
11. Multiply each number in Problem 10 by 100.
12. Multiply each number in Problem 10 by 1000.

6-2. Density of Rational Numbers

One of the observations you have made about the integers is that every integer is preceded by a particular integer, and is followed by a particular integer. The integer which precedes $^{-}8$ is $^{-}9$, and the integer which follows 1005 is 1006. In other words, if n is an integer, then its predecessor is $(n - 1)$, and its successor is $(n + 1)$.

This means that on the number line there are wide gaps between the points which correspond to the integers.

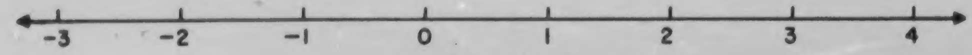


Figure 6-2A

Now consider all the rational numbers, and the points on the number line which correspond to them. Such points are called rational points. On the number line below are shown the rational points between $^{-}3$ and 4 which may be named by the fractions with denominators 2, 3, 4, and 6.

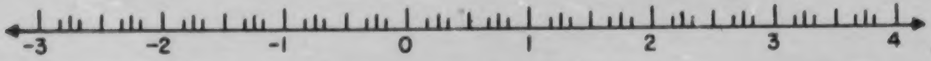


Figure 6-2B

Exercises 6-2a

1. Make a drawing of a number line similar to the one in Figure 6-2A. Mark the points which correspond to these numbers:
$^-(\frac{6}{2})$, $^-(\frac{5}{2})$, $^-(\frac{4}{2})$, $^-(\frac{3}{2})$, $^-(\frac{2}{2})$, $^-(\frac{1}{2})$, $\frac{0}{2}$, $\frac{1}{2}$, $\frac{2}{2}$, $\frac{3}{2}$.

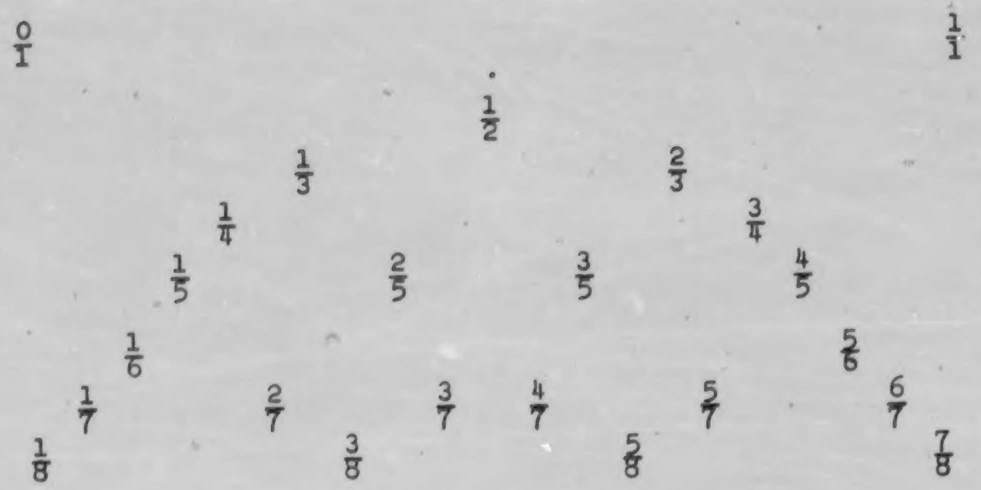
2. On the number line which you drew in Problem 1, find the points that correspond to these numbers:
$^-(\frac{7}{6})$, $^-(\frac{6}{6})$, $^-(\frac{5}{6})$, $^-(\frac{4}{6})$, $^-(\frac{3}{6})$, $^-(\frac{2}{6})$, $^-(\frac{1}{6})$, $\frac{0}{6}$, $\frac{1}{6}$, $\frac{2}{6}$, $\frac{3}{6}$

3. Were any of the points in Problems 1 and 2 the same point? If so, which ones?

4. Suppose you have already located points for the rational numbers which are represented by fractions with denominators 2, 3, 4, 5, and 6. You then locate points represented by fractions with denominator 7. How many new points (not already located) for sevenths will there be between the points for the integers 1 and 2? Between the points for 3 and 4?

5. Suppose that you then locate points for fractions with denominator 8. How many new points will there be between the points for any two consecutive integers?

6. Consider all rational points from 0 to 1 which are named by fractions with denominators 1 to 8 inclusive. These points are named on the following page. The first row shows the fractions with denominator 1, the second row the fraction for the new point with denominator 2, the third row the fractions for the new points with denominator 3, and so on.



$\frac{0}{1}$
$\frac{1}{1}$
$\frac{1}{2}$
$\frac{1}{3}$
$\frac{2}{3}$
$\frac{1}{4}$
$\frac{3}{4}$
$\frac{1}{5}$
$\frac{2}{5}$
$\frac{3}{5}$
$\frac{4}{5}$
$\frac{1}{6}$
$\frac{5}{6}$
$\frac{1}{7}$
$\frac{2}{7}$
$\frac{3}{7}$
$\frac{4}{7}$
$\frac{5}{7}$
$\frac{6}{7}$
$\frac{1}{8}$
$\frac{3}{8}$
$\frac{5}{8}$
$\frac{7}{8}$

(a) Why is $\frac{0}{3}$ omitted from the row for thirds?

(b) Why is $\frac{2}{4}$ omitted from the row for fourths?

(c) Why are there more new points named in the row for fifths and in the row for sevenths than in the row for sixths?

7. The rational numbers named in Problem 6 are combined in one row below, and listed in order from smallest to largest.

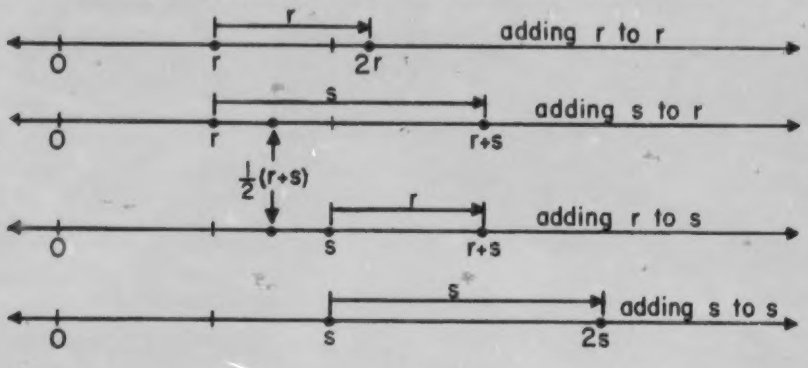
$$\frac{0}{1}\quad \frac{1}{8}\quad \frac{1}{7}\quad \frac{1}{6}\quad \frac{1}{5}\quad \frac{1}{4}\quad \frac{2}{7}\quad \frac{1}{3}\quad \frac{3}{8}\quad \frac{2}{5}\quad \frac{3}{7}\quad \frac{1}{2}\quad \frac{4}{7}\quad \frac{3}{5}\quad \frac{5}{8}\quad \frac{2}{3}\quad \frac{5}{7}\quad \frac{3}{4}\quad \frac{4}{5}\quad \frac{5}{6}\quad \frac{6}{7}\quad \frac{7}{8}\quad \frac{1}{1}$$

Explain why the first six fractions should be in the order shown; the last six fractions.

8. In Problem 7 more numbers could be obtained in the row of fractions and more points among the corresponding set of rational points by inserting next the fractions with denominator 9, then the fractions with denominator 10, and so on. How many new points would correspond to fractions with denominator 9? With denominator 10? With denominator 11?

9. In Problems 6 and 8, which denominator accounts for the largest number of points not already named? What kind of a number seems to account for the largest number of new points when it is used as a denominator? Why?

We may follow a different method for naming and locating new rational points. Consider two positive rational numbers r and s, with $r < s$. Then consider what happens when we add r and s to each of these numbers. Let us consult the number line.



We see that $2r < r + s < 2s$. Taking half of each we get $r < \frac{1}{2}(r + s) < s$. It is not difficult to show that $r < \frac{1}{2}(r+s) < s$, even if r is negative or r and s are both negative. You might try to prove this yourself using the number line, if you wish. The number $\frac{1}{2}(r + s)$ is the average of the numbers r and s. We have observed, then, that the average of two rational numbers is between these numbers. On the number line what point do you suppose corresponds to the average of two numbers? It is the mid-point of the segment determined by the two numbers. If r and s are rational numbers, is $\frac{1}{2}(r + s)$ a rational number? What properties of the rational number system tells us that it is?

We can summarize what we have observed: *The mid-point of the segment joining two rational points on the number line is a rational point corresponding to the average of the two numbers.*

The mid-point of the segment joining the points for $\frac{1}{3}$ and $\frac{1}{2}$ is the point corresponding to the number $\frac{5}{12}$, since

$$\frac{1}{2}\left(\frac{1}{3} + \frac{1}{2}\right) = \frac{1}{2}\left(\frac{3}{6} + \frac{2}{6}\right) = \frac{1}{2} \cdot \frac{5}{6} = \frac{5}{12}\ .$$

The mid-point of the segment joining the points for $\frac{1}{8}$ and $\frac{1}{7}$ is the point corresponding to the number $\frac{15}{112}$, since

$$\frac{1}{2}\left(\frac{1}{8} + \frac{1}{7}\right) = \frac{1}{2}\left(\frac{7}{56} + \frac{8}{56}\right) = \frac{15}{112}$$

By finding the average in this manner it is possible to find rational numbers between each pair of consecutive numbers represented in the row of fractions in Problem 7 of Exercises 6-2a. If we insert these new fractions the row would begin

$$\frac{0}{1},\ \frac{1}{16},\ \frac{1}{8},\ \frac{15}{112},\ \frac{1}{7},\ \frac{13}{84},\ \ldots$$

If you found all the new fractions in this row which could be found in this way, there would be 43 fractions between $\frac{0}{1}$ and $\frac{1}{1}$. This process could be continued indefinitely. You could find points between $\frac{0}{1}$ and $\frac{1}{16}$, between $\frac{1}{16}$ and $\frac{1}{8}$, and so on. You could find as many rational numbers as you wish between 0 and 1 by taking averages, averages of averages, and so on indefinitely.

The discussion above suggests an important property of the rational numbers. This is the

Property of *density*: *Between any two distinct rational numbers there is a third rational number*. On the number line, this means that the number of rational points on any segment is unlimited; no matter how many points on a very small segment have been named, it is possible to name as many more as you please.

Exercises 6-2b

1. Are the integers dense? That is, is there always a third integer between any two integers? Illustrate your answer.
2. Is there a smallest positive integer? a largest?
3. Is there a smallest negative integer? a largest?
4. Is there a smallest positive rational number? a largest negative rational number?
5. Think of the points for 0 and $\frac{1}{100}$ on the number line. Name the rational point P which is halfway between 0 and $\frac{1}{100}$. Name the point halfway between the point P and 0; between the point P and $\frac{1}{100}$.
6. In the same way, find three rational numbers between $\frac{1}{20}$ and $\frac{1}{10}$.
7. Think of the segment with end-points $\frac{1}{1000}$ and $\frac{2}{1000}$. Show a plan you could follow to name as many rational points as you please on this segment. Use your plan to name at least five points.

6-3. Decimal Representations for the Rational Numbers

It is often very helpful to be able to express rational numbers as decimals. When it is necessary to compare two rationals that are very close together, converting to decimal form makes the comparison easier. The decimal form is particularly helpful if there are several rational numbers to be arranged in order. For example, consider the fractions $\frac{13}{25}$, $\frac{27}{50}$, $\frac{3}{8}$, and $\frac{9}{20}$ and their corresponding decimals 0.52, 0.54, 0.375, and 0.45. It is much easier to order the numbers when they are written in decimal form.

Some rational numbers are easily written in decimal form. We know how to write, by inspection,

$\frac{1}{2} = 0.5$, $\frac{1}{4} = 0.25$, $\frac{1}{8} = 0.125$, $\frac{1}{5} = 0.2$, $\frac{1}{25} = 0.04$,

$\frac{1}{125} = 0.008$, and also $\frac{17}{2} = 8.5$, $5\frac{3}{4} = 5.75$, $\frac{175}{10} = 17.5$

For other rational numbers, a decimal expression may not be as obvious but we can always obtain it by the usual process of division. For example

$$\frac{1}{3} = 0.33333\ \ldots$$

$$\frac{8}{3} = 2.6666666\ \ldots$$

$$\frac{1}{7} = 0.142857142857142857\ \ldots$$

$$\frac{1}{13} = 0.07692307692307\ \ldots$$

$$\frac{1}{11} = 0.09090909\ \ldots$$

$$\frac{123}{14} = 8.7857142857142\ \ldots$$

The examples we have discussed seem to suggest that the decimal expansions for rational numbers either terminate (like $\frac{1}{2} = 0.5$) or repeat (like $\frac{1}{3} = 0.3333333\ \ldots$). What would be a reasonable way to study such decimal expansions? Since we have used the division of numerator by denominator to obtain a decimal representation, we might study carefully the process which we carry out in such cases.

Consider the rational number $\frac{7}{8}$. If we carry out the indicated division we write

$$\begin{array}{rl}
.875 & \\
8\overline{)7.000} & \\
\underline{6\ 4} & \\
60 & \text{remainder } 6 \\
\underline{56} & \\
40 & \text{remainder } 4 \\
\underline{40} & \\
0 & \text{remainder } 0
\end{array}$$

In dividing by 8, the only remainders which can occur are 0, 1, 2, 3, 4, 5, 6, and 7. The only remainders which *did* occur are 6 at the first stage, then 4 and finally 0. We could continue dividing, getting at each new stage a remainder of zero and a quotient of zero. We could write $\frac{7}{8} = .875000 \ldots$ but we seldom do. The decimal .875000... is a repeating decimal with 0 repeating over and over again. When the remainder 0 occurs, the division is exact. We say a division is exact if the process of division produces a zero remainder and thereafter zeros as quotients. Such a decimal is often spoken of as a terminating decimal instead of a repeating decimal, and we shall do so at times in this chapter.

What about a rational number which does not have a terminating decimal representation? Suppose we look at a particular example of this kind, say $\frac{2}{13}$. The process of dividing 2 by 13 proceeds like this:

```
        0.153846153
   13 )2.000000000
        1 3               remainder
         70                   7
         65
          50                  5
          39
          110                11
          104
            60                6
            52
             80               8
             78
              20              2
              13
               70             7
               65
                50            5
                39
                11           11
                etc.
```

Here the possible remainders are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12. Not all the remainders do appear, but 7, 5, 11, 6, 8, and 2 occur first in this order. At the next stage in the division the remainder 7 re-occurs and the sequence of remainders 7, 5, 11, 6, 8, and 2 occurs again. In fact the process repeats itself again and again. The corresponding sequence of digits in the quotient--153846--will therefore repeat regularly in the decimal expansion for $\frac{2}{13}$. A repeating decimal is sometimes referred to as a periodic decimal.

In order to write such a periodic decimal concisely and without ambiguity it is customary to write

$$0.153846153846153846 1 \ldots \quad \text{as} \quad 0.\overline{153846} \ldots$$

The bar (vinculum) over the digit sequence 153846 indicates the set of digits which repeats. Similarly, we write 0.3333... as $0.\overline{3}\ldots$. If it seems more convenient we can write 0.3333... as $0.3\overline{3}\ldots$ or $0.33\overline{3}\ldots$, and $0.\overline{153846}\ldots$ as $0.153846\overline{153846}\ldots$

The method we have discussed is quite a general one and it can be applied to any rational number $\frac{a}{b}$. If the indicated division is performed then the only possible remainders which can occur are 0, 1, 2, 3, ... (b - 1). We look only at the stages which contribute to the digits that repeat in the quotient. These stages usually occur after the zeros begin to repeat in the dividend. If the remainder 0 occurs, the decimal expansion terminates at this stage in the division process. Actually, we may write a terminating decimal expansion like 0.25 with a repeated zero to provide a periodic expansion, as 0.25000..., or we may use the bar, as $0.25\overline{0}\ldots$. Note that a zero remainder may occur prior to this stage without terminating the process; for example:

$$\begin{array}{rc}
112.2 & \\
5\overline{)561.0} & \\
\underline{5} & \text{remainder} \\
06 & 0 \\
\underline{5} & \\
11 & 1 \\
\underline{10} & \\
10 & 1 \\
\underline{10} & \\
0 & 0
\end{array}$$

If 0 does not occur as a remainder after zeros are annexed to the dividend, then after at most (b - 1) steps in the division process one of the possible remainders 1, 2, ..., (b - 1) will occur again and the digit sequence will start repeating.

We see from this argument *that any rational number has a decimal expansion which is periodic*.

Exercises 6-3

1. Find decimals for these rational numbers. Continue the division until the repeating begins, and write your answer to at least ten decimal places.

   (a) $\frac{9}{4}$

   (b) $\frac{5}{24}$

   (c) $\frac{3}{7}$

   (d) $\frac{3}{35}$

   (e) $\frac{1}{41}$

   (f) $\frac{128}{125}$

   (g) $\frac{14}{37}$

   (h) $\frac{11}{909}$

   (i) $\frac{1}{82}$

   *(j) $\frac{1}{17}$

2. Which of the following convert to decimals that repeat zero (terminate)?

   (a) $\frac{1}{2}$

   (b) $\frac{1}{3}$

   (c) $\frac{1}{4}$

   (d) $\frac{1}{5}$

   (e) $\frac{1}{6}$

   (f) $\frac{1}{7}$

   (g) $\frac{1}{8}$

   (h) $\frac{1}{9}$

   (i) $\frac{1}{10}$

   (j) $\frac{1}{11}$

   (k) $\frac{1}{12}$

   (l) $\frac{1}{13}$

3. Write in completely factored form the denominators of those fractions that terminated in Problem 2.

4. Carry to six decimal places the following fractions.

   (a) $\frac{1}{7}$

   (b) $\frac{2}{7}$

   (c) $\frac{3}{7}$

   (d) $\frac{4}{7}$

   (e) $\frac{5}{7}$

   (f) $\frac{6}{7}$

6-4. The Rational Number Corresponding to a Periodic Decimal

We saw how to find by division the decimal expansion of a given rational number. We have found that the decimal expansion is periodic. But suppose we have the opposite situation, that is, we are given a periodic decimal. Does such a decimal in fact represent a rational number? How can we find out?

We can see how to approach this problem by considering an example. Let us write the number $0.132132132132\ldots$ and call it n, so that $n = 0.132132\overline{132}\ldots$. The periodic block of digits is 132, so we multiply by 1000 which shifts the first block of digits to the left of the decimal point and gives the relation

$$\begin{array}{r} 1000n = 132.132132\overline{132}\ldots \\ \underline{\quad n = \quad 0.132132\overline{132}\ldots} \end{array}$$

By subtracting we obtain $$999n = 132$$

so that $$n = \frac{132}{999}$$

or, in simplest form $$n = \frac{44}{333}.$$

We find by this process that $0.132132132\overline{132}\ldots = \frac{44}{333}$.

The example here illustrates a general procedure which mathematicians have developed to show that every periodic decimal *represents a rational number*. We see, therefore, *that there is a one-to-one correspondence between the set of rational numbers, and the set of periodic decimals*. It would be quite equivalent, then, for us to define the rational numbers as the set of numbers represented by all such periodic decimals.

Before we leave the subject of decimals we want to discuss one interesting fact about terminating decimals.

We saw that rationals like $\frac{1}{2} = 0.5$, $\frac{1}{5} = 0.2$, $\frac{15}{8} = 1.875$, $\frac{397}{1000} = 0.397$, $\frac{692}{25} = 27.68$ are all represented by terminating decimals. How can we determine when this will be the case? If, for inspiration, we look at the rationals of this type which we discussed, we see an obvious clue: the denominators seem to have only the prime factors 2 or 5, or both. (See Problem 3 in Exercises 6-3.)

Consider a rational number in which the denominator is a power of 2, such as $\frac{39}{2^4}$.

By multiplying by $\frac{5^4}{5^4}$, or 1, we can write $\frac{39 \cdot 5^4}{2^4 \cdot 5^4}$.

Since $2^4 \cdot 5^4$ may be written

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 5 \cdot 5 = (2 \cdot 5) \cdot (2 \cdot 5) \cdot (2 \cdot 5) \cdot (2 \cdot 5) = 10^4,$$

we can write:

$$\frac{39}{2^4} = \frac{39 \cdot 5^4}{2^4 \cdot 5^4} = \frac{39 \cdot 5^4}{10^4} = \frac{39 \cdot 625}{10,000} = \frac{24,375}{10,000} = 2.4375$$

Similarly if we have a rational number in which the denominator is a power of 5, we can proceed as in the following example,

$$\frac{3}{3125} = \frac{3}{5^5} = \frac{3 \cdot 2^5}{5^5 \cdot 2^5} = \frac{3 \cdot 32}{10^5} = \frac{96}{100,000} = 0.00096.$$

Can you show that $5^5 \cdot 2^5 = 10^5$?

Quite generally, if we have any rational number with only powers of 2 and powers of 5 in the denominator, we can use the same technique. For example,

$$\frac{3791}{2^7 \cdot 5^4} = \frac{3791 \cdot 5^7 \cdot 2^4}{(2^7 \cdot 5^4)(5^7 \cdot 2^4)} = \frac{3791 \cdot 5^7 \cdot 2^4}{(2^7 \cdot 5^7)(5^4 \cdot 2^4)} = \frac{3791 \cdot 5^7 \cdot 2^4}{10^7 \cdot 10^4} = \frac{3791 \cdot 5^7 \cdot 2^4}{10^{11}}$$

and this gives a terminating decimal representation. Can you prove that $\frac{3791}{2^7 \cdot 5^4}$ is a terminating decimal by multiplying by $\frac{5^3}{5^3}$?

In order to establish a general fact of this kind suppose we ask the following question. What rational number $\frac{p}{q}$ (p and q assumed to have only 1 as a common factor) can be represented by $\frac{N}{10^k}$ where N is an integer?

Suppose that this is indeed the case and that

$$\frac{p}{q} = \frac{N}{10^k} .$$

Therefore $$q \cdot N = p \cdot 10^k.$$

This says that q divides the product of p and 10^k. But we assumed that p and q have only 1 as a common factor. Hence q must divide 10^k. But the only possible factors of 10^k are numbers which are powers of 2 multiplied by powers of 5.

Thus we have proved that a rational number r has a terminating decimal representation if and only if the denominator of r consists only of the product of a power of 2 and a power of 5. Then r must be of the form

$$r = \frac{p}{2^m \, 5^n}$$

where p is an integer.

Class Exercise 6-4

1. Perform each of the following subtractions.
   - (a) $10n - n$
   - (b) $100n - n$
   - (c) $1,000n - 10n$
   - (d) $100n - 10n$
   - (e) $1,000n - n$
   - (f) $10,000n - 100n$

2. Write the products as a single number.
   - (a) $10 \times 0.99\overline{9}\ldots$
   - (b) $100 \times 3.12\overline{12}\ldots$
   - (c) $1,000 \times 0.035\overline{035}\ldots$
   - (d) $10 \times 16.66\overline{6}\ldots$
   - (e) $10 \times 0.004\overline{4}\ldots$
   - (f) $1,000 \times 0.61345\overline{345}\ldots$
   - (g) $100 \times 8.03151\overline{5}\ldots$
   - (h) $100 \times 312.899\overline{9}\ldots$
   - (i) $10 \times 312.899\overline{9}\ldots$
   - (j) $10,000 \times 6.01230\overline{123}\ldots$

3. Subtract in each of the following.
   - (a) $$\begin{array}{r} 3128.99\overline{9}\ldots \\ \underline{312.89\overline{9}}\ldots \end{array}$$
   - (b) $$\begin{array}{r} 9.99\overline{9}\ldots \\ \underline{0.99\overline{9}}\ldots \end{array}$$
   - (c) $$\begin{array}{r} 162.162\overline{162}\ldots \\ \underline{0.162\overline{162}}\ldots \end{array}$$
   - (d) $$\begin{array}{r} 301.0101\overline{01}\ldots \\ \underline{3.0101\overline{01}}\ldots \end{array}$$
   - (e) $$\begin{array}{r} 1.23333\overline{3}\ldots \\ \underline{0.12333\overline{3}}\ldots \end{array}$$
   - (f) $$\begin{array}{r} 354.54\overline{54}\ldots \\ \underline{3.54\overline{54}}\ldots \end{array}$$
   - (g) $$\begin{array}{r} 27075.07\overline{5075}\ldots \\ \underline{27.07\overline{5075}}\ldots \end{array}$$
   - (h) $$\begin{array}{r} 416.4777\overline{7}\ldots \\ \underline{41.6477\overline{7}}\ldots \end{array}$$

4. For each of the following numbers N find the smallest number of the form 10^k (10, 100, 1000, etc.) so that $(10^k \cdot N)-N$ is a terminating decimal. Show this to be true.

   Example:

$$\begin{array}{rr} N = & 1.324\overline{24}\ldots \\ 100N = & 132.424\overline{24}\ldots \\ N = & 1.324\overline{24}\ldots \\ \hline 100N - N = & 131.10000\ldots \end{array}$$

   - (a) $0.55\overline{5}\ldots$
   - (b) $0.73\overline{73}\ldots$
   - (c) $0.901\overline{901}\ldots$
   - (d) $3.0233\overline{3}\ldots$
   - (e) $163.17\overline{7}\ldots$
   - (f) $672.42\overline{42}\ldots$
   - (g) $0.1234565\overline{6}\ldots$
   - (h) $3.4100\overline{0}\ldots$

5. Express each of the following in the form $\frac{a}{b}$, where a and b are counting numbers.

   (a) $\frac{3.1}{99}$

   (b) $\frac{4.11}{9}$

   (c) $\frac{16.3}{99}$

   (d) $\frac{1.03}{999}$

   (e) $\frac{382.4}{9}$

   (f) $\frac{47.531}{9999}$

6. If a is replaced by 1 in $\frac{a}{b}$, by what numbers between 23 and 50 may b be replaced so that $\frac{a}{b}$ can be represented by a terminating decimal?

Exercises 6-4

1. What rational numbers in form $\frac{a}{b}$ have these decimal expressions?

   (a) $0.0\overline{909}$ ...

   (b) $0.111\overline{1}$ ...

   (c) $0.055\overline{5}$ ...

   (d) $0.123\overline{123}$ ...

   (e) 0.1625

   (f) $0.166\overline{6}$ ...

   (g) $5.125\overline{125}$ ...

   (h) $10.045\overline{45}$ ...

2. Write each denominator of the following numbers in completely factored form.

   (a) $\frac{7}{32}$

   (b) $\frac{47}{100}$

   (c) $\frac{5}{9}$

   (d) $\frac{13}{50}$

   (e) $\frac{12}{35}$

   (f) $\frac{21}{80}$

   (g) $\frac{71}{120}$

   (h) $\frac{1}{160}$

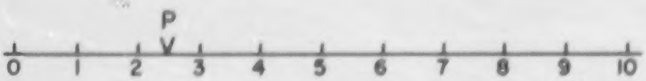
3. Which of the numbers in Problem 2 have decimals which repeat zero?

4. If a is replaced by 1 in the rational number $\frac{a}{b}$, by what numbers between 63 and 101 may b be replaced so that $\frac{a}{b}$ can be represented by a terminating decimal expression?

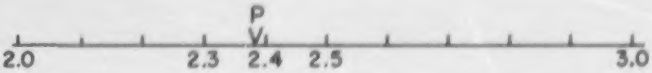
6-5. Rational Points on the Number Line

If we use decimal representations for rational numbers, we see immediately how to locate and how to order the corresponding points on the number line.

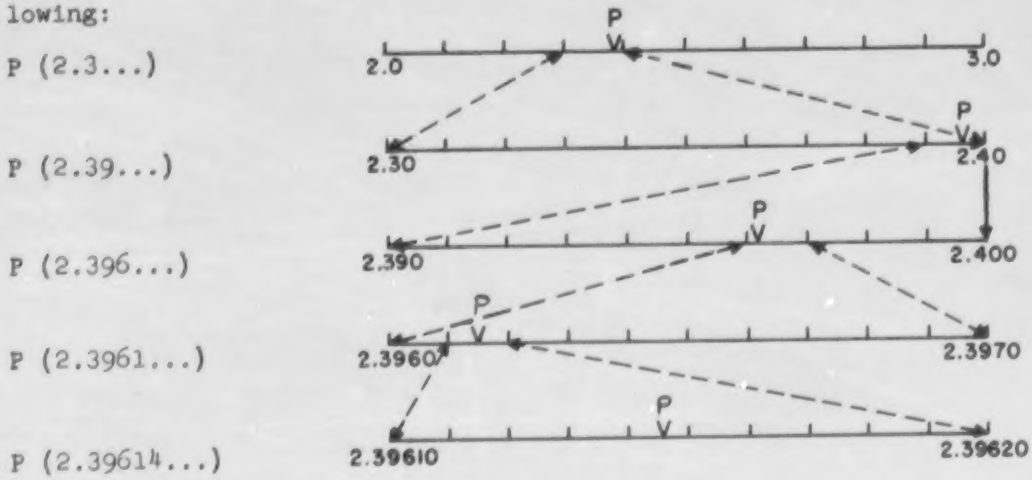
Consider for example the rational number $2.39\overline{614}\ldots$ and its place on the number line. The digit 2 in the units place tells us immediately that the corresponding rational point P lies between the integers 2 and 3 on the number line. Graphically then the first rough picture is this:



A more precise description is obtained by looking at the first two digits 2.3 which tell us immediately that P lies between 2.3 and 2.4. On the interval from 2 to 3, then, divided into tenths (and magnified ten times for easy comparison) we find P as shown below



If we continue the process of successively refining the location of P on the number line we have a picture such as the following:



Location of point P corresponding to $2.39\overline{614}$...

From such a decimal representation for a rational number we easily find how to locate the number to any desired degree of accuracy on the number line.

Moreover, given any two distinct rational numbers in this form it is a simple matter to tell by inspection which is larger and which is smaller, and which precedes the other on the number line.

If you think of locating the point $\frac{3}{7}$ carefully on the number line would you prefer to use $\frac{3}{7}$ or $0.\overline{428571}$ ...? If you wish to compare $\frac{3}{7}$ with another rational, which form is easier to use, $\frac{3}{7}$ or $0.\overline{428571}$ ...?

Exercises 6-5

1. Arrange each group of decimals in the order in which the points to which they correspond would occur on the number line. List first the point farthest to the left.

(a) 1.379 1.493 1.385 5.468 1.372
(b) $^{-}9.426$ $^{-}2.765$ $^{-}2.761$ $^{-}5.630$ $^{-}2.763$
(c) $^{-}0.15475$ 0.15467 0.15463 0.15475 0.15598

2. In Problem (1c), which points lie on the following segments:
   (a) The segment with endpoints 1 and 2?
   (b) The segment with endpoints 0 and 1?
   (c) The segment with endpoints 0.1 and 0.2?
   (d) The segment with endpoints 0.15 and 0.16?
   (e) The segment with endpoints 0.154 and 0.155?

3. Draw a 10 centimeter segment; label the endpoints 0 and 1, and divide the segment into tenths. Mark and label the following points:

   (a) 0.23 (b) 0.49 (c) 0.80 (d) 0.6 (e) 0.08 (f) 0.95

4. Arrange each group of rational numbers in order of increasing size by first expressing them in decimal form.

   (a) $\frac{3}{9}$, $\frac{4}{10}$, $\frac{17}{50}$ (c) $\frac{3}{7}$, $\frac{4}{9}$

   (b) $\frac{2}{3}$, $\frac{7}{10}$, $\frac{67}{100}$ (d) $\frac{152}{333}$, $\frac{415}{909}$

6-6. Irrational Numbers

We have learned many things about rational numbers. One of the most important is the density property; between any two distinct rational numbers on the number line there is a third rational number. This tells us that there are many rational numbers and rational points--*very* many of them. Moreover, they are spread throughout the number line. Any segment, no matter how small, contains infinitely many rational points. One might think that *all* the points on the number line are rational points. Let us locate a certain point on the number line by a very simple compass and straight edge construction. Perhaps this point will have a surprise for us.

(a) Draw a number line and call it ℓ. Let A be the point zero and B be the point one.

(b) At B, draw a ray m perpendicular to ℓ.

(c) On m draw a line segment $\overline{BC}$, one unit long.

(d) Draw segment $\overline{AC}$.

(e) With A as center and radius AC, draw a circular arc which intersects ℓ. Call the point of intersection D.

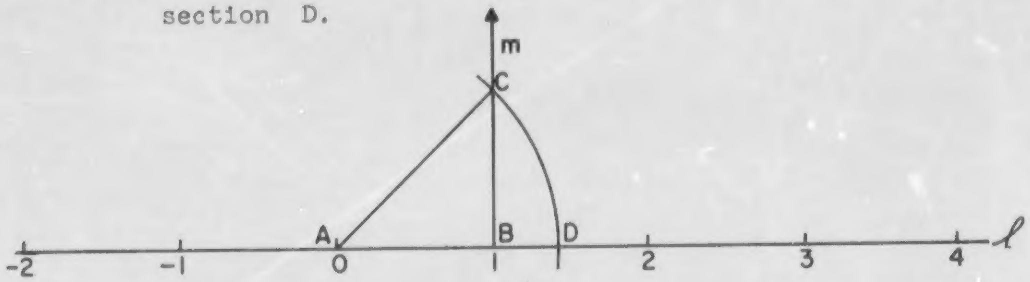


Figure 6-6

Now consider two questions:

(1) To what number (if any) does point D correspond?

(2) Is this number a rational number?

Consider the first question, "To what number does point D correspond?" First find the length of $\overline{AC}$, since $\overline{AC}$ and $\overline{AD}$ have the same length. We shall use as unit of measure the unit distance on the number line. In Figure 6-6, triangle ABC is a right triangle. The measure of $\overline{AB}$ is 1. The measure of $\overline{BC}$ is 1. We can use the Pythagorean property to find AC.

$$(AC)^2 = (BC)^2 + (AB)^2$$

$$(AC)^2 = 1^2 + 1^2$$

$$(AC)^2 = 2$$

The positive number whose square is 2 is defined as the square root of 2 and is written $\sqrt{2}$.

Thus,

$$AC = \sqrt{2}, \text{ so}$$

$$AD = \sqrt{2}.$$

Therefore, the point D corresponds to the number $\sqrt{2}$. Is $\sqrt{2}$ a rational number? Is it the quotient of two integers, and can it be represented as a fraction $\frac{p}{q}$, in which p and q are integers and $q \neq 0$?

To answer this question, we shall follow a line of reasoning which people very often use. It is the type of reasoning which John's mother used one day when John was late from school. When his mother scolded him he said that he had run all the way home. "No, you didn't run all the way," she said firmly. John was ashamed, and asked, "How did you know?" "If you had run all that way, you would have been out of breath," she said. "You are not out of breath. Therefore you did not run."

John's mother had used indirect reasoning. She assumed the opposite of the statement she wished to prove, and showed that this assumption led to a conclusion which could not possibly be true. Therefore her assumption had to be false, and the original statement had to be true.

We shall prove that $\sqrt{2}$ is not a rational number. We use indirect reasoning. We shall assume that $\sqrt{2}$ is a rational number and show that this assumption leads to an impossible conclusion.

Assume that $\sqrt{2}$ is a rational number. Then we can write $\sqrt{2}$ as $\frac{p}{q}$, where p and q are integers and $q \neq 0$. Take $\frac{p}{q}$ in simplest form. This means that p and q have no common factor except 1.

If $\sqrt{2} = \frac{p}{q}$, then $2 = \frac{p^2}{q^2}$, and so $2q^2 = p^2$. Since p and q are integers, then p^2 and q^2 are also integers. If $p^2 = 2q^2$ then p^2 must be an even number. (An integer is even if it is equal to 2 times another integer.) Thus, $p \cdot p$ must be even. An odd number times an odd number is an odd number. (Do you remember why?) Thus, p must be even, and can be written as $2a$, where a is an integer.

Then $p^2 = 2q^2$ may be written as

$$\begin{aligned} & & (2a)^2 &= 2q^2 \\ &\text{and} & (2a)\cdot(2a) &= 2q^2 \\ &\text{and} & 2\cdot(2a^2) &= 2q^2 \\ &\text{and} & 2a^2 &= q^2 \end{aligned}$$

This tells us that q^2 is also an even number since it is equal to 2 times another integer. So q is also an even number.

Thus our assumption, that $\sqrt{2}$ is a rational number $\frac{p}{q}$ in simplest form, has led us to the conclusion that p and q both have the factor 2. This is impossible, since the simplest form for a fraction is the one in which p and q have no common factor other than 1. So the statement "$\sqrt{2}$ is a rational number" must be false.

Since the measure of segment $\overline{AD}$ in Figure 6-6 is $\sqrt{2}$, then $\sqrt{2}$ must be the number which corresponds to point D. It has been shown that $\sqrt{2}$ is not a rational number. Therefore, there is at least this one point on the number line which corresponds to some number which is not a rational number. In other words, even though the rational points are dense, the set of points on the number line contains more points than there are rational numbers.

A number like $\sqrt{2}$, which is not a rational number, is called an irrational number. The prefix "ir" changes the meaning of "rational" to "not rational."

Exercises 6-6

1. Construct a figure like Figure 6-6, and label point D "$\sqrt{2}$". Then use your compass to locate the point which corresponds to the number $^-(\sqrt{2})$, and label it.

2. Draw a number line, using a unit of the same length as the unit in Problem 1. Use the letter A for the point 0 and the letter B for the point 2. At B construct a segment perpendicular to the number line and 1 unit in length, and call it $\overline{BP}$. Draw $\overline{AP}$. What is the measure of segment $\overline{AP}$?

3. Use the drawing for Problem 2, and locate on the number line the points which correspond to $\sqrt{5}$ and $^{-}(\sqrt{5})$. Label the points.

4. Do you think $\sqrt{5}$ is a rational number or an irrational number? Why?

*5. Using the same method as in Problems 2 and 3, locate the point $\sqrt{3}$. Can you work out a way to locate the point for $\sqrt{6}$? For $\sqrt{7}$?

6. Locate the points which correspond to these numbers?

   (a) $2\sqrt{2}$ (b) $3\sqrt{2}$ (c) $^{-}(3\sqrt{2}.)$

7. Do you think that $(2\sqrt{2})$ is a rational number or an irrational number?

8. BRAINBUSTER: Prove that $\sqrt{5}$ is an irrational number. (use indirect reasoning very similar to the line of reasoning which we used to show that $\sqrt{2}$ is irrational. At one point you will have to know that if p^2 has 5 as a factor, then p also has 5 as a factor. Prove this simple fact. Before you try to prove that $\sqrt{5}$ is irrational, think of the unique factorization property of counting numbers. If the prime number 5 were not a factor of p then how could it be a factor of p^2?)

In the preceding discussion it was proved that $\sqrt{2}$ is not a rational number. It is a great surprise to find that we can so easily construct a line segment whose length is not given by a rational number. Moreover, it appears that there are many other numbers, such as $\sqrt{3}$ and $\sqrt{5}$ which are not rationals. If you think about the rationals and the irrationals a bit you can see how to write many, many irrationals. For example, every number of the form $\frac{a}{b}\sqrt{2}$, where $\frac{a}{b}$ is rational, will be irrational. Hence the set $\{\frac{a}{b}\sqrt{2}\}$ can be put into one-to-one correspondence with the set of rationals $\{\frac{a}{b}\}$. Yet the set $\{\frac{a}{b}\sqrt{2}\}$ is obviously only a very small part of the irrationals!

Indeed, we have suffered a great disillusionment--the rational numbers, despite being dense on the number line, actually leave empty more positions than they fill!

When we set up a one-to-one correspondence between a given set and the set of counting numbers (or a subset of the set of counting numbers) mathematicians say we have "enumerated" the set. We can "enumerate" the set of all rational numbers, but Georg Cantor (1845-1918) discovered in 1874 that the set of irrational numbers cannot be "enumerated" by any method. There are so many irrational numbers that it is impossible to set up a one-to-one correspondence between the set of these numbers and the set of counting numbers. No matter how you try to display irrational numbers some irrational numbers will always be left out; more will be left out than have been included, as a matter of fact. This is what we mean when we say that the rational numbers leave more places empty on the number line than they fill.

If you are interested in learning more about this important phase of mathematics you might refer to One Two Three ... Infinity by George Gamow (pages 14-23). A brief but interesting history of Cantor's life can be found in Men of Mathematics by E. T. Bell (Chapter 29).

6-7. A Decimal Representation for $\sqrt{2}$

Numbers like $\sqrt{5}$ and $\sqrt{2}$ correspond to points on the number line; they specify lengths of line segments and they satisfy our natural notion of what a number is. Perhaps the most unusual aspect about $\sqrt{2}$ is the way it was defined: $\sqrt{2}$ is the positive number n which when squared yields 2, so that

$$n^2 = 2.$$

This differs from our previous way of defining numbers, since up to now we have dealt mainly with integers and numbers defined as ratios of integers.

In order to help us gain a better understanding of $\sqrt{2}$ we shall look for a new way of describing $\sqrt{2}$ in terms of more familiar notions. If, for example, we could somehow express $\sqrt{2}$ as a decimal this would help us to compare it with the rational numbers we know. It would also tell us where to place it on the number line.

Let us think about the definition of the number $\sqrt{2}$, namely $(\sqrt{2})^2 = 2$. If we think of squaring 1 and 2 we note immediately that

$$1^2 < (\sqrt{2})^2 < 2^2 \text{ and hence } 1 < \sqrt{2} < 2.$$

This tells us that $\sqrt{2}$ is greater than 1 and less than 2, but we already know that. We might try a closer approximation by testing the squares of 1.1, 1.2, 1.3, 1.4, 1.5. A little arithmetic of this sort (try it!) leads us to the result

$$1.96 = (1.4)^2 < (\sqrt{2})^2 < (1.5)^2 = 2.25,$$

and therefore we conclude that $1.4 < \sqrt{2} < 1.5$.

The arithmetic involves slightly more work at the next stage but we see that

$$1.9881 = (1.41)^2 < (\sqrt{2})^2 < (1.42)^2 = 2.0164,$$

and therefore

$$1.41 < \sqrt{2} < 1.42.$$

If we try to extend the process further we get at the next stage

$$1.414 < \sqrt{2} < 1.415.$$

You can see that this process can be continued as long as our enthusiasm lasts, and gives a better decimal approximation at every stage. If we continued to 7 place decimals we would find

$$1.4142135 < \sqrt{2} < 1.4142136$$

This is a very good approximation of $\sqrt{2}$, for

$$(1.4142135)^2 = 1.99999982358225,$$

and

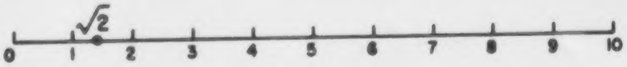
$$(1.4142136)^2 = 2.00000010642496.$$

By the use of the defining property, $(\sqrt{2})^2 = 2$, then, we can find decimal approximations for $\sqrt{2}$ which are as accurate as we wish. We are led to write

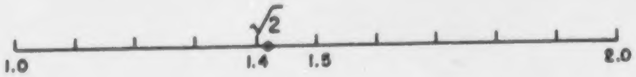
$$\sqrt{2} = 1.4142135\ .\ .\ .$$

where the three dots indicate that the digits continue without terminating, as the process above suggests.

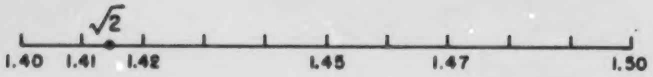
Geometrically the procedure we have followed can be described as follows in the number line. Looking first at the integers of the number line on the segment from 0 to 10, we saw that $\sqrt{2}$ would be between 1 and 2.



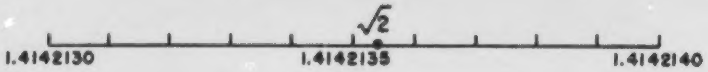
Enlarging our view of this segment (by a ten-fold magnification) we saw that $\sqrt{2}$ is on the segment with end-points 1.4 and 1.5



and again magnifying this picture, $\sqrt{2}$ lies within the interval (1.41, 1.42)



and so on till the 8th stage shows us that $\sqrt{2}$ lies between 1.4142135 and 1.4142136.



This process shows us how to read the successive digits in the decimal representation for $\sqrt{2}$. At the same time it gives a way to define the position of the point on the real line.

When we write the number $\sqrt{2}$ as $1.4142135\ \ldots$ it looks suspiciously like many rational numbers we have seen, such as

$$\frac{1}{3} = 0.3333333\ \ldots \quad \text{and} \quad \frac{1}{7} = 0.14285714\ \ldots\ .$$

We pause to ask, how are they different and how can we tell a rational from an irrational number when we see only the decimal representations of the numbers?

The one special feature of the decimal representation of a rational number is that it is a periodic decimal. As we have seen, every periodic decimal represents a rational number. Then the decimal representation of $\sqrt{2}$ cannot be periodic, for $\sqrt{2}$ is irrational. We can be sure that as we continue to find new digits in the decimal representation,

$$\sqrt{2} = 1.4142135\ \ldots,$$

no group of digits will ever repeat indefinitely. We can only be certain that a decimal names a rational number when the period of the decimal is indicated, usually with a vinculum ($\overline{}$).

Exercises 6-7

1. Between what two consecutive integers are the following irrational numbers? (Write your answer as suggested for (a)). Use the table on pages 200-201.

   (a) $\sqrt{30}$ $[\underline{\ ?\ } < \sqrt{30} < \underline{\ ?\ }]$

   (b) $\sqrt{89}$

   (c) $\sqrt{253}$

   (d) $\sqrt{4280}$ (Hint: 4280 is 42.80×10^2, so begin estimating by thinking of $\sqrt{43} \cdot 10$)

   (e) $\sqrt{9315}$

2. Express (a), (b), and (c) as decimals to six places.

   (a) $(1.731)^2$

   (b) $(1.732)^2$

   (c) $(1.733)^2$

   (d) Find the difference between your answer for (a) and the number 3; find the difference between your answer for (b) and the number 3; find the difference between your answer for (c) and the number 3.

   (e) To the nearest thousandth what is the best decimal expression for $\sqrt{3}$?

Which of the numbers suggested is the better approximation of the following irrational numbers?

3. $\sqrt{3}$: 1.73 or 1.74

4. $\sqrt{15}$: 3.87 or 3.88

5. $\sqrt{637}$: 25.2 or 25.3

Find, to the nearest tenth, the nearest decimal expression for these irrational numbers:

6. $\sqrt{10}$
7. $\sqrt{149}$
8. $\sqrt{221}$

*9. For what positive number n (to one decimal place) is $n^2 = 10$?

*10. For what positive number n (to one decimal place) is $n^2 = 149$?

6-8. Irrational Numbers and the Real Number System

We have seen that all rational numbers have periodic decimal representations and that all periodic decimals correspond to rational numbers. We saw also that $\sqrt{2}$ is not rational and that therefore, it is represented by a non-periodic decimal. Hence we have called $\sqrt{2}$ an irrational number.

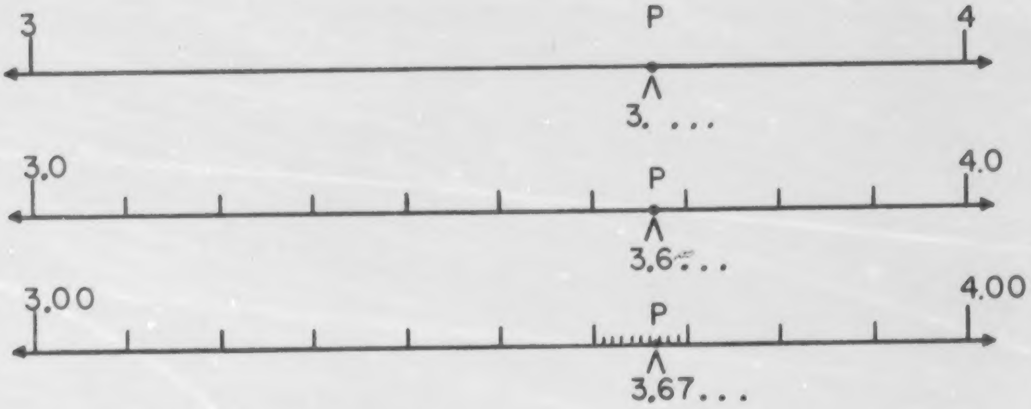
We now use this decimal form to define the set of irrational numbers. *We define an irrational number as any number with a non-periodic decimal representation*.

The system composed of all rational and irrational numbers we call the *real number system*.

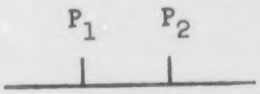
From this we see that *any real number can be characterized by a decimal representation*.

If the decimal representation is periodic the number is a rational number, otherwise the number is an irrational number.

With every point P on the real number line we associate one and only one number of this form by a process of successive locations in decimal intervals of decreasing length. The drawings on the following page illustrate the first few steps in finding the decimal corresponding to a point P on the number line. Consider point P between 3 and 4.



Note that any two distinct points P_1 and P_2 will correspond to distinct decimal representations, for if they occur as



on the number line we need only subdivide the number line by a sufficiently fine decimal subdivision (tenths, hundredths, thousandths, etc.) to assure that P_1 and P_2 are separated by a point of subdivision.

Conversely, given any decimal, we have found how to locate the corresponding point of the real number line by considering successive rational decimal approximations provided by the number. (Remember how we started to locate the point $2.39\overline{61}\ldots$ in Section 6-5.)

Thus there is a one-to-one correspondence between the set of real numbers and the set of points on the number line.

The set of real numbers contains the set of rational numbers as a subset. We have learned that these rational numbers form a mathematical system with operations, addition and multiplication and their inverses, subtraction and division. The same is true of the entire set of real numbers. We can add real numbers, rational or irrational, and we can multiply real numbers. The resulting number system has all of the properties of the rational number system. In addition it has one important property which the rational number system does not have. This will be discussed below.

Before we list these properties we should pause to ask what we know about the operations themselves. You should not have much trouble understanding the meaning of addition in the real number system in terms of the number line. Even though there is no simpler name for a sum such as $\sqrt{2} + \sqrt{3}$ than the symbol "$\sqrt{2} + \sqrt{3}$" itself, you can think of a method of constructing the point $\sqrt{2} + \sqrt{3}$ on the number line by placing segments of length $\sqrt{2}$ and $\sqrt{3}$ end to end.

The meaning of multiplication is somewhat harder to illustrate. Given segments of length $\sqrt{2}$ and $\sqrt{3}$ it is possible to describe a geometric construction of a point which we would naturally call $\sqrt{2} \cdot \sqrt{3}$. However, you will have to study Chapter 9 before you will be prepared to understand such a construction. The two operations can also be given meaning in terms of the decimal representation which we have described, but here, too, difficulties are encountered which you are not yet ready to handle. This should not cause you undue concern. Even a mathematician often has to accept things which he does not fully understand in order to get on with the work which is of immediate interest to him. But if these ideas he accepts are important, he always returns to them as soon as he can and masters them. You will return to the real number system again as you study mathematics in the future, and each time you will understand more of the definition and the meaning of the operations.

We list first the familiar properties which the real number system shares with the rational number system.

Property 1. Closure

(a) Closure under Addition. The real number system is closed under the operation of addition, i.e., if a and b are real numbers then $a + b$ is a real number.

(b) Closure under Subtraction. The real number system is closed under the operation of subtraction (the inverse of addition), i.e., if a and b are real numbers then $a - b$ is a real number.

(c) Closure under Multiplication. The real number system is closed under the operation of multiplication, i.e., if a and b are real numbers then $a \cdot b$ is a real number.

(d) Closure under Division. The real number system is closed under the operation of division (the inverse of multiplication), i.e., if a and b are real numbers then $a \div b$ (when $b \neq 0$) is a real number.

The operations of addition, subtraction, multiplication, and division on real numbers display the properties which we have already observed for rationals. These may be summarized as follows:

Property 2. Commutativity

(a) If a and b are real numbers, then $a + b = b + a$.

(b) If a and b are real numbers, then $a \cdot b = b \cdot a$.

Property 3. Associativity

(a) If a, b, and c are real numbers, then $a + (b + c) = (a + b) + c$.

(b) If a, b, and c are real numbers, then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

Property 4. Identities

(a) If a is a real number, then $a + 0 = a$, i.e., zero is the identity element for the operation of addition.

(b) If a is a real number, then $a \cdot 1 = a$, i.e., one is the identity element for the operation of multiplication.

Property 5. Distributivity

If a, b, and c are real numbers, then $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

Property 6. Inverses

(a) If a is a real number, there is a real number (^{-}a), called the additive inverse of a such that $a + (^{-}a) = 0$.

(b) If a is a real number and $a \neq 0$ there is a real number b, called the multiplicative inverse of a such that $a \cdot b = 1$.

Property 7. Order

The real number system is ordered, i.e., if a and b are different real numbers then either $a < b$ or $a > b$.

Property 8. Density

The real number system is dense, i.e., between any two distinct real numbers there is always another real number. Consequently, between any two real numbers we find as many more real numbers as we wish. In fact we easily see that: (1) There is always a rational number between any two distinct real numbers, no matter how close. (2) There is always an irrational number between any two distinct real numbers, no matter how close. (You will learn how to illustrate this later in this section.)

The ninth property of the system of real numbers is one which is not shared by the rationals.

Property 9. Completeness

The real number system is complete, that is, not only does a point on the number line correspond to each real number, but conversely, a real number corresponds to each point on the number line.

The rational numbers differ from the real numbers in this respect. A point on the number line corresponds to each rational number, but no rational number corresponds to certain points on the number line. We have seen that in the system of rationals there is no number $\sqrt{2}$ which when squared yields 2. However, in the real number system as defined, such a number is included.

If a and b are positive rational numbers and $b = a^n$ we write

$$a = \sqrt[n]{b}$$

(read "a is an nth root of b").

It happens that the nth root of any positive rational number which is not itself the nth power of a rational number is an irrational number. This means that such numbers as,

$$\sqrt{3}, \quad \sqrt[3]{\frac{5}{17}}, \quad \sqrt[4]{15}, \sqrt{\frac{2}{3}},$$

are irrational numbers, whereas,

$$\sqrt{25}, \quad \sqrt[4]{16}, \sqrt{\frac{81}{144}}, \quad \sqrt[3]{8},$$

are rational numbers. Hence, in the system of rational numbers we cannot hope to extract nth roots of any rational numbers which are not nth powers of rational numbers. However, when we introduce the irrationals to form the real number system we can find nth roots of all positive rational numbers, and of all positive real numbers as well. Thus a very useful property of the real number system is:

> The real number system contains nth roots, $\sqrt[n]{\frac{a}{b}}$ of all positive rational numbers $\frac{a}{b}$, $b \neq 0$; the real number system contains nth roots of all positive real numbers.

This assures us that we can find among the real numbers such numbers as

$$\sqrt{3}, \quad \sqrt{7}, \quad \sqrt[3]{4}, \quad \sqrt[5]{23}$$

and any other nth roots of positive rational numbers, as well as numbers which are sums, differences, products, and quotients of such numbers. For example:

$$1 + \sqrt{5}, \quad \sqrt[3]{4} - \sqrt[4]{5}, \quad \sqrt{2} \cdot \sqrt{3}, \text{ and } \frac{\sqrt{5}}{\sqrt{7}}$$

are real numbers.

In addition to irrational numbers which arise from finding roots of rational numbers there are many more irrational numbers which are called transcendental numbers. One example of a transcendental number is the number π which you have met in your study of circles. Recall that π is the ratio of the measure of the circumference of a circle to the measure of its diameter. It was surprisingly hard to prove that π is irrational, but it has been done. The decimal representation

$$\pi = 3.14159265 \ldots$$

is not repeating. The number π is not $\frac{22}{7}$, although $\frac{22}{7}$ is a fair approximation to π. (Compare the decimal representation of $\frac{22}{7}$ with that of π.)

When you study logarithms in high school, you will be studying numbers that, with a few exceptions, are transcendental numbers. If N is any positive real number and x is the exponent such that

$$10^x = N$$

then we say that x is the logarithm of N to the base 10. If N is a power of 10, say $N = 10^2$, then clearly $10^x = 10^2$, so 2 is the logarithm of 10^2 to the base 10. In such a case, the logarithm is a rational number. But for most numbers the logarithm will be a (transcendental) irrational number.

The trigonometric ratios, such as the sine and the tangent of an angle, are other expressions which usually turn out to be transcendental irrational numbers. These ratios are defined in Chapter 9.

Exercises 6-8a

1. Which of the following numbers do you think are rational and which irrational? Make two lists.

   (a) $0.231\overline{231}\ldots$

   (b) $0.23123112311123\ldots$

   (c) $\frac{3\sqrt{2}}{7\sqrt{2}}$

   (d) $\sqrt{7}$

   (e) $0.78\overline{342}\ldots$

   (f) $\frac{\pi}{2}$

   (g) $\frac{3}{4}\sqrt{6}$

   (h) $9 - \sqrt{3}$

   (i) $0.7500\overline{0}\ldots$

   (j) $\frac{58}{11}$

   (k) $0.959559555955559\ldots$

2. Write each of the rational numbers in Problem 1 as a decimal and as a fraction.

3. For each of the irrational numbers in Problem 1 write a decimal correct to the nearest hundredth.

4. (a) Write three terminating decimals for rational numbers.

   (b) Write three repeating decimals for rational numbers.

   (c) Write three decimals for irrational numbers.

You have learned how to insert other rational numbers between two given rationals. Now that you have studied decimal representation for real numbers, you can see how to insert either rational or irrational numbers between real numbers. Look at these decimals for two numbers a and b.

$$a = 4.219317317317\overline{317}\ldots$$

$$b = 4.236565565556555556\ldots$$

These numbers are quite close together, but any decimal which begins 4.22 ... will be greater than a and less than b. We can then continue the decimal in such a way as to make it rational or to make it irrational. We can even make the decimal terminating if we wish to do so. For example, 4.222 and $4.225\overline{225}\ldots$ are rational numbers while 4.225622566225666... is irrational. All of these numbers lie between a and b.

In order to be sure that a real number written in the decimal form is an irrational number we must make certain that it is not periodic. An easy way to do this is to repeat a number of one or more digits and to follow this repeating number by a different digit once, then twice, then three times, and so on. For example in the number $b = 4.2365655655565555\ldots$ the digit 6 is repeated and is followed by a 5, once, twice, and so on. In the number 4.237823788237888... the number 237 is repeated and is followed by 8 once, then twice, then three times, and so on. Because the digit 8 occurs an increasing number of times in the formation of this decimal, the decimal cannot have a definite period. The number represented must therefore be irrational.

Exercises 6-8b

1. (a) Write a decimal for a rational number between $2.3846\overline{846}\ldots$ and $2.369\overline{369}\ldots$.

   (b) Write a decimal for an irrational number between the numbers in (a).

2. Write decimals for (a) a rational number and (b) an irrational number between 0.346019... and 0.342806...

3. Write decimals for (a) a rational number and (b) an irrational number between 67.283... and 67.28106006...

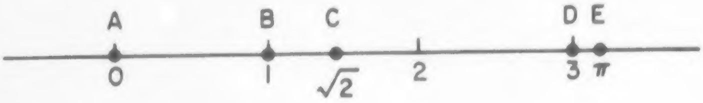
4. Do you think that the real number system contains square roots of all integers? Support your answer by an example.

5. An approximation which the Babylonians used for π was the interesting ratio $\frac{355}{113}$. How good an approximation is this? Is it as good as $\frac{22}{7}$?

6-9. Geometric Properties of the Real Number Line

The one-to-one correspondence between the real numbers and the points of the number line gives us for the first time a satisfactory geometric representation of numbers. For this reason it is customary to refer to the number line as *the real number line*.

We know that there are no gaps or missing points in the real number line. We can speak of tracing the real number line continuously and know that the segment described at any stage has a length which is measured by a real number. Thus in the number line indicated below



we know that $\overline{BC}$ has a length of measure $\sqrt{2} - 1$, the length of $\overline{CD}$ has measure $3 - \sqrt{2}$, $\overline{BE}$ has measure $\pi - 1$, $\overline{CE}$ is measured by $\pi - \sqrt{2}$.

We can think of a point moving continuously from 0 to 1. At every location we may associate with it a real number.

Because of this continuous property of our real number system, we sometimes refer to it as *the continuum of real numbers*.

Rational Approximations to Irrationals

Whenever we give an irrational number in its decimal form, for example, $N = 0.019234675\ldots$, we see that we automatically define a sequence of rational numbers which give closer and closer approximations to the irrational number N. We can read such a sequence of rational approximations as,

0.01
0.019
0.0192
0.01923
0.019234
0.0192346
0.01923467
0.019234675

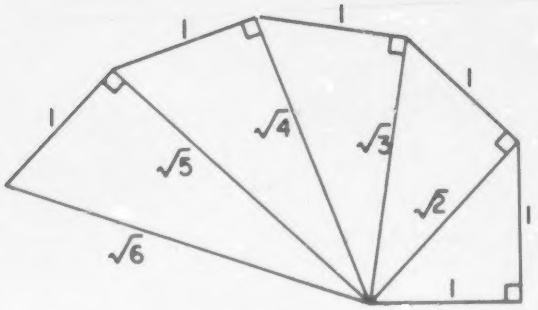
Rationals and Irrationals in the World Around Us

We see many examples of rationals every day--the price of groceries, the amount of a bank balance, the rate of pay, the amount of a weekly salary, the grade on a test paper.

Although we have not considered the irrationals for very long, it is easy to see many examples which involve irrational numbers. For example, consider a circle of radius one unit. What is its circumference? Why 2π units, of course. In fact, any circle whose radius is a rational number has a circumference which is irrational. Also, the circular closed region of radius r has an area, the measure of which is an irrational number (πr^2) whenever r is rational and usually whenever r is irrational.

The volume of a circular cylinder is found by the formula $V = \pi r^2 h$ and its lateral surface area A by $A = 2\pi rh$ where h is the altitude of the cylinder. Here also the volume and area are given by irrational numbers if the radius r and altitude h are given as rationals.

Also, we learn how to draw some lengths of irrational measure by the following simple succession of right triangles:



You will notice however that this process gives some rational lengths since $\sqrt{4} = 2$, $\sqrt{9} = 3$, and so on.

Draw this succession of right angles on your paper. Continue until you have drawn a length to represent $\sqrt{10}$.

Exercises 6-9

1. Which of the following numbers are rational and which are irrational?

   The number of units in:

   (a) the circumference of a circle whose radius is $\frac{1}{2}$ unit.

   (b) the area of a square whose sides are one unit long.

   (c) the hypotenuse of a right triangle whose sides are 5 and 12 units long.

   (d) the area of a square whose sides have length $\sqrt{3}$ units.

   (e) the volume of a cylinder whose height is 2 units and whose base has radius 1 unit.

   (f) the area of a right triangle with hypotenuse of length 2 units and equal sides.

2. With the use of the facts that $\sqrt{2} \approx 1.414$ and that $\sqrt{3} \approx 1.732$ show that $\sqrt{2} \cdot \sqrt{3} \approx \sqrt{6} \approx 2.449$.

3. When we begin to compute with irrational numbers we sometimes encounter relationships which look rather peculiar at first but which make perfect sense on closer inspection.
   Here are two examples:
   The multiplicative inverse of $\sqrt{2}$ is $\frac{1}{2}\sqrt{2}$.
   The multiplicative inverse of $(\sqrt{3} + \sqrt{2})$ is $(\sqrt{3} - \sqrt{2})$.
   *(a) Verify these assertions approximately by using the decimal approximations given in Problem 2.
   (b) BRAINBUSTER: Verify these assertions exactly by computing with the irrational numbers themselves.

*4. Find the radius of a circle whose circumference is 2. Give an approximate value for the radius. (Use 3.14 for π.)

INDEX

The reference is to the page on which the term occurs.

accuracy, 223
addition, 3, 12, 18
    property, 76, 91
additive inverse, 18, 19, 38, 47, 237
altitude
    of cone, 484
    of pyramid, 477
angle(s)
    complementary, 365
    corresponding, 159
Antarctic Circle, 529
antipodal points, 515
antipode, 515
apex of pyramid, 476
approximation, 276
arc, 162
Arctic Circle, 529
area, 145
    of circle, 458
    of regular polygon, 457
    of surface of sphere, 535, 536, 538
    of trapezoid, 454
associativity, 236, 270
average, 243
axes, 24
axis of symmetry, 172
base, 472
    of pyramid, 476
bisect, 164
bisector, 164
    perpendicular, 191
British Imperial gallon, 153
Cantor, Georg, 262
capacity, 150
centimeter, 139
chance events, 311
circle, 64, 170, 171, 173, 175, 511
    Antarctic, 529
    Arctic, 529
    area of, 64, 458
    circumference of, 64
    great, 516, 520
    interior of, 520
    semi-, 517
    small, 516, 517
        length of, 540

- closed number phrase, 53
- closure, 236, 270
- combinations, 303
- commutativity, 236, 270
- compass, 157
- complementary angles, 365
- completeness, 271
- compound sentence, 68
- concurrent lines, 167
- condition, 30
- cone, 484
  - altitude of, 484
  - height of, 484
  - right circular, 484
  - slant height of, 484
  - vertex of, 484
- congruent, 180
  - triangles, 176
- conjecture, 549
  - Goldbach, 549
- constant, 397
  - of proportionality, 397
- construction(s), 160, 163
- continuum of real numbers, 276
- coordinates, 21, 23, 24
- corresponding angles, 159
- cosine, 359
- cotangent, 367
- counting number, 3, 235
- cube, 467
- cubic meter, 148
- cylinder
  - right circular, 469
- Dantzig, 572, 573
- decimal
  - expansion, 246
  - non-periodic, 267
  - periodic, 248, 250, 265, 267, 275
  - point, 134
  - repeating, 247
  - representation, 246, 265, 267, 273
  - terminating, 247
- decimeter, 139
- dekometer, 139
- density, 240, 244, 271
- Descartes, 569
- diameter, 514
- digit
  - significant, 220, 231, 232
    - in the product, 231
- dimensional, 421
- direct variation, 396
- directed segment, 3, 12

distance
    between parallel planes, 464
    from a point to a plane, 464
distributivity, 236, 271
division, 42, 128
edges, 466
Egyptian, 193
empirical probability, 323
enumerate, 262
equation(s), 58, 60
    equivalent, 75
    linear, 97
equality, 76, 83
equivalent equations, 75
error
    greatest possible, 216, 219, 228
    percent of, 223
    relative, 223
Euclid, 162
Euler, 447, 549
Euler's formula, 447, 569
event(s)
    independent, 335
    mutually exclusive, 330, 342
exponent, 121, 126, 129, 149, 273
    negative, 121, 126, 129
    notation, 134
    zero as, 121
factorial, 296, 297
fiducial point, 3
formula, 60
Franklin, 563
Fulkerson, 572, 573
Galileo, 407
gallon, 153
    British Imperial, 153
Goldbach Conjecture, 549
grade, 377
gram, 150
graph, 30
gravitation
    Newton's law of, 410
great circle, 516
greatest possible error, 216, 219
    of a sum, 228
Greenwich meridian, 527
half-line, 3, 22
Heawood, 563, 569
hectometer, 139
height
    of cone, 484
    of pyramid, 477
hexagon, 170

hexagonal
    pyramid, 477
    right prism, 210, 469
Hilbert, David, 577, 578
hyperbola, 405
hypotenuse, 194, 195
identity, 236, 270
inch, 143
independent event, 335
indirect reasoning, 259
inequality, 58, 91
integer(s), 3, 9, 225
interior
    of a circle, 520
    of a sphere, 520
inverse, 271
    additive, 237
    multiplicative, 237
    variation, 401
irrational number, 257, 260, 267, 275
Johnson, 572, 573
kilometer, 139
lateral
    edge
        of oblique prism, 472
        of pyramid, 477
        of right prism, 466
    face(s)
        of oblique prism, 472
        of pyramid, 476
        of right prism, 466
latitude, 529
    parallels of, 517, 526
laws of variation, 394
Lehmer, D. N., 546
length, 138
    of small circle, 540
light year, 118
line(s)
    concurrent, 167
    half-, 3, 22
    parallel, 158
    perpendicular, 158
    real number, 276
linear equation, 97
liter, 150
logarithm, 273
longitude, 528
mass, 150
median, 169
mega-, 141
meridian
    Greenwich, 527
    zero, 527

- meter, 138
  - square, 145
- metric
  - system, 137, 154
  - ton, 151
- micro-, 141
- micron, 141
- millimeter, 139
- multiplication, 34, 125
  - property, 83
- multiplicative inverse, 237
- mutually exclusive events, 330, 342
- negative
  - exponent, 121, 126, 129
  - number, 8, 9
- Newton's law of gravitation, 410
- non-periodic decimal, 267
- notation
  - exponent, 134
  - powers-of-ten, 115, 133
- number(s)
  - counting, 3, 235
  - irrational, 257, 260, 267, 275
  - line, 1, 2, 3, 5, 255, 276
    - real, 276
  - negative, 8, 9
  - phrase, 52
    - closed, 53
    - open, 53
  - positive, 3
  - prime, 546
  - rational, 9, 255
  - real, 267
    - continuum of, 276
    - number line, 276
    - number system, 276
  - sentence, 51
  - transcendental, 273
- oblique prism, 471, 472
- octagon, 170
- one-to-one correspondence, 2, 262, 268, 276
- open
  - phrase, 53
  - sentence, 59
- opposite, 9, 13, 19
- order, 237, 271
- ordered pair, 25, 33, 94
- origin, 3, 21
- pantograph, 158
- parabola, 103, 407
- parallel(s)
  - lines, 158
  - of latitude, 517, 526
  - ruler, 158

parallelogram, 206
Pascal, 286
Pascal triangle, 286
pentagon, 170
percent of error, 223
periodic decimal, 248, 250, 265, 275
permutation, 292
    symbol, 295
perpendicular, 190
    bisector, 191
    line, 158
    to the plane, 462
phrase, 52
    number, 52
        closed, 53
        open, 53
pi (π), 64, 232, 273, 275, 535
plane(s)
    distance between parallel, 464
    distance from point to, 464
    perpendicular to, 462
plotting, 25
point(s)
    antipodal, 515
    fiducial, 3
polygon
    area of regular, 457
    regular, 170
    simple closed, 434
polygonal path, 434
polyhedron, 428
positive number, 3
power, 272
powers-of-ten notation, 115, 133
precision, 219, 223
prime number, 546
prism
    hexagonal right, 210, 469
    lateral edge of
        oblique, 472
        right, 466
    lateral face of
        oblique, 472
        right, 466
    oblique, 471, 472
    rectangular right, 208
    right, 208
    triangular right, 468
probability, 312, 313, 341
    empirical, 323
    of A and B, 334
    of A or B, 328

property
    addition, 76, 91
    multiplication, 83
    Pythagorean, 193, 202
protractor, 158
pyramid, 210
    altitude of 477
    apex of, 476
    base of, 476
    height of, 477
    hexagonal, 477
    lateral edge of, 477
    lateral face of, 476
    regular, 478
    slant height of, 480
    square, 477
Pythagoras, 193
Pythagorean property, 193, 202
quadrants, 27
quadrilateral, 170, 206
radius, 163
Rankin, 559
rational number, 9, 255
real number(s), 267
    continuum of, 276
    line, 276
    system, 276
reasoning
    indirect, 259
rectangle, 206
rectangular
    right prism, 208
    system, 24
regular
    polygon, 170
    pyramid, 478
relative error, 223
repeating decimal, 247
rhombus, 452
right
    circular cone, 484
    circular cylinder, 469
    prism, 208
    triangle, 192
root, 272
ruler
    parallel, 158
scale, 386
scientific notation, 114, 221
segment
    directed, 3, 12
selections, 303
    symbol, 303, 304

semicircle, 517
sentence
  compound, 68
  number, 51
  open, 59
set of solutions, 60
significant digit, 220, 232
  in the product, 231
similar triangles, 380
simple
  closed polygon, 434
  surface, 442
simplex, 422
sin, 366
sine, 359
slant height
  of cone, 484
  of pyramid, 480
slope, 374
small circle, 516, 517
solid
  spherical, 533
solution set, 60
sphere, 511
  interior of, 520
  surface area of, 535, 536, 538
  tangent to, 515
  volume of, 533, 535, 542
spherical
  soap bubbles, 537
  solid, 533
square(s)
  meter, 145
  pyramid, 477
  table of, 200-201
square root, 196
  table of, 200-201
straightedge, 157
subtraction, 46
surface
  of a sphere, 533, 535, 538
  simple, 442
symmetry, 172
  axis of, 172
table
  of square roots, 200-201
  of squares, 200-201
  of trigonometric ratios, 368
tangent, 358
  to the sphere, 515
tetrahedron, 418, 477
terminating decimal, 247
tolerance, 217

topological, 564
transcendental number, 273
transversal, 159
trapezoid, 206, 454
    area of, 454
triangle(s)
    congruent, 176
    Pascal, 286
    right, 192
    similar, 380
triangular right prism, 468
trigonometric ratios
    table of, 368
Tropic
    of Cancer, 529
    of Capricorn, 529
T-square, 158
twin primes, 552
variation, 392
    direct, 396
    inverse, 401
    laws of, 394
varies
    directly, 397
    inversely, 414
vertex of cone, 484
vertices, 466
vinculum, 248, 265
Vinogradov, 549
volume, 148
    of sphere, 533, 535, 542
weight, 152
    of water, 152
Williams, 573
zero
    as exponent, 121
    meridian, 527